

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1

MATHEMATICS 1001-004

WINTER 2024

SOLUTIONS

[5] 1. (a) Let $u = \ln(x)$ so $du = \frac{1}{x} dx$. The integral becomes

$$\begin{aligned}\int \frac{\ln^2(x)}{x} dx &= \int u^2 du \\ &= \frac{u^3}{3} + C \\ &= \frac{1}{3} \ln^3(x) + C.\end{aligned}$$

[6] (b) We use integration by parts with $w = \ln(x)$ so $dw = \frac{1}{x} dx$, and $dv = \frac{1}{x^2} dx$ so $v = -\frac{1}{x}$. Thus

$$\begin{aligned}\int \frac{\ln(x)}{x^2} dx &= -\frac{\ln(x)}{x} + \int \frac{1}{x^2} dx \\ &= -\frac{\ln(x)}{x} - \frac{1}{x} + C.\end{aligned}$$

[4] (c) Since a basic property of logarithms is that $\ln(x^y) = y \ln(x)$, we can rewrite the given integral as

$$\begin{aligned}\int \frac{\ln(x^2)}{\ln(x)} dx &= \int \frac{2 \ln(x)}{\ln(x)} dx \\ &= 2 \int dx \\ &= 2x + C.\end{aligned}$$

[25] 2. (a) First we write

$$\int \frac{4-x}{\sqrt{4-x^2}} dx = 4 \int \frac{1}{\sqrt{4-x^2}} dx - \int \frac{x}{\sqrt{4-x^2}} dx.$$

The first integral is an elementary arcsine integral:

$$\int \frac{1}{\sqrt{4-x^2}} dx = \arcsin\left(\frac{x}{2}\right) + C.$$

For the second integral, we let $u = 4 - x^2$ so $du = -2x dx$ and $-\frac{1}{2} du = x dx$. Then

$$\int \frac{x}{\sqrt{4-x^2}} dx = -\frac{1}{2} \int u^{-\frac{1}{2}} du = -\frac{1}{2} [2\sqrt{u}] + C = -\sqrt{4-x^2} + C.$$

Hence we have

$$\begin{aligned} \int \frac{4-x}{\sqrt{4-x^2}} dx &= 4 \arcsin\left(\frac{x}{2}\right) - \left(-\sqrt{4-x^2}\right) + C \\ &= 4 \arcsin\left(\frac{x}{2}\right) + \sqrt{4-x^2} + C. \end{aligned}$$

(b) Since we can write

$$\int \frac{\cosh(x)}{\sinh(x)\sqrt{9\sinh^2(x)-1}} dx = \int \frac{\cosh(x)}{\sinh(x)\sqrt{[3\sinh(x)]^2-1}} dx,$$

we let $u = 3\sinh(x)$ so $du = 3\cosh(x) dx$ and $\frac{1}{3} du = \cosh(x) dx$. Furthermore, this means that $\sinh(x) = \frac{1}{3}u$ and so the integral becomes

$$\begin{aligned} \int \frac{\cosh(x)}{\sinh(x)\sqrt{9\sinh^2(x)-1}} dx &= \int \frac{1}{\frac{1}{3}u\sqrt{u^2-1}} \cdot \frac{1}{3} du \\ &= \int \frac{1}{u\sqrt{u^2-1}} du \\ &= \operatorname{arcsec}(u) + C \\ &= \operatorname{arcsec}(3\sinh(x)) + C. \end{aligned}$$

(c) Using long division of polynomials, we have

$$\begin{array}{r} 3x^2 - 4x \\ 2x + 3 \overline{) 6x^3 + x^2 - 12x + 5} \\ \underline{6x^3 + 9x^2} \\ -8x^2 - 12x + 5 \\ \underline{-8x^2 - 12x} \\ 5 \end{array}$$

Thus we can write the integral as

$$\begin{aligned} \int \frac{6x^3 + x^2 - 12x + 5}{2x + 3} dx &= \int \left(3x^2 - 4x + \frac{5}{2x + 3} \right) dx \\ &= 3 \cdot \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + 5 \cdot \frac{1}{2} \ln|2x + 3| + C \\ &= x^3 - 2x^2 + \frac{5}{2} \ln|2x + 3| + C. \end{aligned}$$

(d) Since

$$\int x^7 \sec^2(x^4) dx = \int x^4 \sec^2(x^4) \cdot x^3 dx,$$

we let $u = x^4$ so $du = 4x^3 dx$ and $\frac{1}{4} du = x^3 dx$. The integral becomes

$$\int x^7 \sec^2(x^4) dx = \frac{1}{4} \int u \sec^2(u) du.$$

Now we use integration by parts with $w = u$ so $dw = du$, and $dv = \sec^2(u) du$ so $v = \tan(u)$. This yields

$$\begin{aligned} \int x^7 \sec^2(x^4) dx &= \frac{1}{4} \left[u \tan(u) - \int \tan(u) du \right] \\ &= \frac{1}{4} [u \tan(u) + \ln|\cos(u)|] + C \\ &= \frac{1}{4} x^4 \tan(x^4) + \frac{1}{4} \ln|\cos(x^4)| + C. \end{aligned}$$

(e) We use integration by parts with $w = x^2$ so $dw = 2x dx$, and $dv = \sin(3x) dx$ so $v = -\frac{1}{3} \cos(3x)$. Hence

$$\int x^2 \sin(3x) dx = -\frac{1}{3} x^2 \cos(3x) + \frac{2}{3} \int x \cos(3x) dx.$$

Now we use integration by parts again, this time with $w = x$ so $dw = dx$, and $dv = \cos(3x) dx$ so $v = \frac{1}{3} \sin(3x)$. We obtain

$$\begin{aligned} \int x^2 \sin(3x) dx &= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{3} \left[\frac{1}{3} x \sin(3x) - \frac{1}{3} \int \sin(3x) dx \right] \\ &= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{9} x \sin(3x) - \frac{2}{9} \int \sin(3x) dx \\ &= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{9} x \sin(3x) - \frac{2}{9} \left[-\frac{1}{3} \cos(3x) \right] + C \\ &= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{9} x \sin(3x) + \frac{2}{27} \cos(3x) + C. \end{aligned}$$

(f) First we complete the square:

$$\begin{aligned}9x^2 - 6x + 17 &= 9 \left[x^2 - \frac{2}{3}x + \frac{17}{9} \right] \\&= 9 \left[\left(x^2 - \frac{2}{3}x + \frac{1}{9} \right) + \frac{17}{9} - \frac{1}{9} \right] \\&= 9 \left[\left(x - \frac{1}{3} \right)^2 + \frac{16}{9} \right] \\&= 9 \left(x - \frac{1}{3} \right)^2 + 16 \\&= 3^2 \left(x - \frac{1}{3} \right)^2 + 16 \\&= (3x - 1)^2 + 16.\end{aligned}$$

Now we let $u = 3x - 1$ so $du = 3 dx$ and $\frac{1}{3} du = dx$. The integral becomes

$$\begin{aligned}\int \frac{1}{9x^2 - 6x + 17} dx &= \frac{1}{(3x - 1)^2 + 16} dx \\&= \frac{1}{3} \int \frac{1}{u^2 + 16} du \\&= \frac{1}{3} \cdot \frac{1}{4} \arctan \left(\frac{u}{4} \right) + C \\&= \frac{1}{12} \arctan \left(\frac{3x - 1}{4} \right) + C.\end{aligned}$$