

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1

MATHEMATICS 1001-003

WINTER 2026

SOLUTIONS

- [6] 1. (a) We use integration by parts with $w = x^2$ so $dw = 2x dx$, and $dv = \cos(3x) dx$ so $v = \frac{1}{3} \sin(3x)$. Thus

$$\int x^2 \cos(3x) dx = \frac{1}{3} x^2 \sin(3x) - \frac{2}{3} \int x \sin(3x) dx.$$

Now we use integration by parts again. This time we let $w = x$ so $dw = dx$, and $dv = \sin(3x) dx$ so $v = -\frac{1}{3} \cos(3x)$. This yields

$$\begin{aligned} \int x^2 \cos(3x) dx &= \frac{1}{3} x^2 \sin(3x) - \frac{2}{3} \left[-\frac{1}{3} x \cos(3x) + \frac{1}{3} \int \cos(3x) dx \right] \\ &= \frac{1}{3} x^2 \sin(3x) + \frac{2}{9} x \cos(3x) - \frac{2}{9} \cdot \frac{1}{3} \sin(3x) + C \\ &= \frac{1}{3} x^2 \sin(3x) + \frac{2}{9} x \cos(3x) - \frac{2}{27} \sin(3x) + C. \end{aligned}$$

- [5] (b) Let $u = x^3$ so $du = 3x^2 dx$ and $\frac{1}{3} du = x^2 dx$. The integral becomes

$$\begin{aligned} \int x^2 \cos(x^3) dx &= \frac{1}{3} \int \cos(u) du \\ &= \frac{1}{3} \sin(u) + C \\ &= \frac{1}{3} \sin(x^3) + C. \end{aligned}$$

- [5] (c) We can rewrite the integral as

$$\begin{aligned} \int \left(\frac{\cos(x) + 3}{\cos(x)} \right)^2 dx &= \int \left(\frac{\cos(x)}{\cos(x)} + \frac{3}{\cos(x)} \right)^2 dx \\ &= \int [1 + 3 \sec(x)]^2 dx \\ &= \int [1 + 6 \sec(x) + 9 \sec^2(x)] dx \\ &= x + 6 \ln|\sec(x) + \tan(x)| + 9 \tan(x) + C. \end{aligned}$$

[12] 2. (a) First we complete the square:

$$\begin{aligned}
 15 - 4x - 4x^2 &= -4 \left[x^2 + x - \frac{15}{4} \right] \\
 &= -4 \left[\left(x^2 + x + \frac{1}{4} \right) - \frac{15}{4} - \frac{1}{4} \right] \\
 &= -4 \left[\left(x + \frac{1}{2} \right)^2 - 4 \right] \\
 &= 16 - 4 \left(x + \frac{1}{2} \right)^2 \\
 &= 16 - (2x + 1)^2.
 \end{aligned}$$

Thus

$$\frac{1}{\sqrt{15 - 4x - 4x^2}} dx = \int \frac{1}{\sqrt{16 - (2x + 1)^2}} dx.$$

Now we let $u = 2x + 1$ so $du = 2 dx$ and $\frac{1}{2} du = dx$. The integral becomes

$$\begin{aligned}
 \frac{1}{\sqrt{15 - 4x - 4x^2}} dx &= \frac{1}{2} \int \frac{1}{\sqrt{16 - u^2}} du \\
 &= \frac{1}{2} \arcsin \left(\frac{u}{4} \right) + C \\
 &= \frac{1}{2} \arcsin \left(\frac{2x + 1}{4} \right) + C.
 \end{aligned}$$

(b) We use integration by parts. One approach is to let $w = e^{2x}$ so $dw = 2e^{2x} dx$, and $dv = \sin(x) dx$ so $v = -\cos(x)$. Then

$$\int e^{2x} \sin(x) dx = -e^{2x} \cos(x) + 2 \int e^{2x} \cos(x) dx.$$

We use integration by parts again. We still let $w = e^{2x}$ so $dw = 2e^{2x} dx$, but now we let $dv = \cos(x) dx$ so $v = \sin(x)$. Thus

$$\begin{aligned}
 \int e^{2x} \sin(x) dx &= -e^{2x} \cos(x) + 2 \left[e^{2x} \sin(x) - 2 \int e^{2x} \sin(x) dx \right] \\
 &= -e^{2x} \cos(x) + 2e^{2x} \sin(x) - 4 \int e^{2x} \sin(x) dx.
 \end{aligned}$$

Now observe that the original integral has reoccurred, so we can write

$$\begin{aligned}
 5 \int e^{2x} \sin(x) dx &= -e^{2x} \cos(x) + 2e^{2x} \sin(x) \\
 \int e^{2x} \sin(x) dx &= -\frac{1}{5} e^{2x} \cos(x) + \frac{2}{5} e^{2x} \sin(x) + C.
 \end{aligned}$$

(c) Observe that we can rewrite the integral as

$$\int x^3 \sqrt{x^2 + 1} dx = \int x^2 \sqrt{x^2 + 1} \cdot x dx.$$

We let $u = x^2 + 1$ so $du = 2x dx$ and $\frac{1}{2} du = x dx$. Then $x^2 = u - 1$, and the integral becomes

$$\begin{aligned} \int x^3 \sqrt{x^2 + 1} dx &= \int (u - 1) \sqrt{u} \cdot \frac{1}{2} du \\ &= \frac{1}{2} \int (u^{\frac{3}{2}} - u^{\frac{1}{2}}) du \\ &= \frac{1}{2} \left[\frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} \right] + C \\ &= \frac{1}{5} (x^2 + 1)^{\frac{5}{2}} - \frac{1}{3} (x^2 + 1)^{\frac{3}{2}} + C. \end{aligned}$$

[12] 3. (a) Observe that we can rewrite the integral as

$$\int \frac{\sec(x) \tan(x)}{9 \sec^2(x) + 25} dx = \int \frac{\sec(x) \tan(x)}{[3 \sec(x)]^2 + 25} dx.$$

Let $u = 3 \sec(x)$ so $du = 3 \sec(x) \tan(x) dx$ and $\frac{1}{3} du = \sec(x) \tan(x) dx$. The integral becomes

$$\begin{aligned} \int \frac{\sec(x) \tan(x)}{9 \sec^2(x) + 25} dx &= \frac{1}{3} \int \frac{1}{u^2 + 25} du \\ &= \frac{1}{3} \cdot \frac{1}{5} \arctan\left(\frac{u}{5}\right) + C \\ &= \frac{1}{15} \arctan\left(\frac{3 \sec(x)}{5}\right) + C. \end{aligned}$$

(b) Since the integrand is an improper rational functions, we use long division of polynomials:

$$\begin{array}{r} x^2 - 3 \\ 2x + 1 \overline{) 2x^3 + x^2 - 6x} \\ \underline{2x^3 + x^2} \\ -6x \\ \underline{-6x - 3} \\ 3 \end{array}$$

Thus we can write the integral as

$$\begin{aligned}\int \frac{2x^3 + x^2 - 6x}{2x + 1} dx &= \int \left(x^2 - 3 + \frac{3}{2x + 1} \right) dx \\ &= \frac{x^3}{3} - 3x + 3 \cdot \frac{1}{2} \ln|2x + 1| + C \\ &= \frac{1}{3}x^3 - 3x + \frac{3}{2} \ln|2x + 1| + C.\end{aligned}$$

- (c) We use integration by parts with $w = \operatorname{arcsec}(x)$ so $dw = \frac{1}{x\sqrt{x^2 - 1}} dx$, and $dv = x dx$ so $v = \frac{1}{2}x^2$. This yields

$$\begin{aligned}\int x \operatorname{arcsec}(x) dx &= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{2} \int \frac{x^2}{x\sqrt{x^2 - 1}} dx \\ &= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{2} \int \frac{x}{\sqrt{x^2 - 1}} dx.\end{aligned}$$

For the remaining integral, let $u = x^2 - 1$ so $du = 2x dx$ and $\frac{1}{2} du = x dx$. We obtain

$$\begin{aligned}\int x \operatorname{arcsec}(x) dx &= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{2} \cdot \frac{1}{2} \int \frac{1}{\sqrt{u}} du \\ &= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{4} \int u^{-\frac{1}{2}} du \\ &= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{4} [2\sqrt{u}] + C \\ &= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{2} \sqrt{x^2 - 1} + C.\end{aligned}$$