MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 | **MATHEMATICS 1001-002** | WINTER 2024

SOLUTIONS

[6] 1. (a) We use integration by parts with $w = \ln(x)$ so $dw = \frac{1}{x}\,dx$, and $dv = \frac{1}{x^2}\,dx$ so $v = -\frac{1}{x}$. Thus

$$\int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln(x)}{x} + \int \frac{1}{x^2}\,dx$$
$$= -\frac{\ln(x)}{x} - \frac{1}{x} + C.$$

[5] (b) Let $u = \ln(x)$ so $du = \frac{1}{x}\,dx$. The integral becomes

$$\int \frac{\ln^2(x)}{x}\,dx = \int u^2\,du$$
$$= \frac{u^3}{3} + C$$
$$= \frac{1}{3}\ln^3(x) + C.$$

[4] (c) Since a basic property of logarithms is that $\ln(x^y) = y\ln(x)$, we can rewrite the given integral as

$$\int \frac{\ln(x^2)}{\ln(x)}\,dx = \int \frac{2\ln(x)}{\ln(x)}\,dx$$
$$= 2\int dx$$
$$= 2x + C.$$

[25] 2. (a) We use integration by parts with $w = x^2$ so $dw = 2x\,dx$, and $dv = \sin(3x)\,dx$ so $v = -\frac{1}{3}\cos(3x)$. Hence

$$\int x^2\sin(3x)\,dx = -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\int x\cos(3x)\,dx.$$

Now we use integration by parts again, this time with $w = x$ so $dw = dx$, and $dv = \cos(3x)\,dx$ so $v = \frac{1}{3}\sin(3x)$. We obtain

$$\int x^2\sin(3x)\,dx = -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\left[\frac{1}{3}x\sin(3x) - \frac{1}{3}\int \sin(3x)\,dx\right]$$
$$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) - \frac{2}{9}\int \sin(3x)\,dx$$
$$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) - \frac{2}{9}\left[-\frac{1}{3}\cos(3x)\right] + C$$
$$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x) + C.$$

(b) Since we can write

$$\int \frac{\cosh(x)}{\sinh(x)\sqrt{9\sinh^2(x)-1}}\,dx = \int \frac{\cosh(x)}{\sinh(x)\sqrt{[3\sinh(x)]^2-1}}\,dx,$$

we let $u = 3\sinh(x)$ so $du = 3\cosh(x)\,dx$ and $\frac{1}{3}\,du = \cosh(x)\,dx$. Furthermore, this means that $\sinh(x) = \frac{1}{3}u$ and so the integral becomes

$$\begin{aligned}
\int \frac{\cosh(x)}{\sinh(x)\sqrt{9\sinh^2(x)-1}}\,dx &= \int \frac{1}{\frac{1}{3}u\sqrt{u^2-1}}\cdot\frac{1}{3}\,du \\
&= \int \frac{1}{u\sqrt{u^2-1}}\,du \\
&= \operatorname{arcsec}(u) + C \\
&= \operatorname{arcsec}(3\sinh(x)) + C.
\end{aligned}$$

(c) First we complete the square:

$$\begin{aligned}
9x^2 - 6x + 17 &= 9\left[x^2 - \frac{2}{3}x + \frac{17}{9}\right] \\
&= 9\left[\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + \frac{17}{9} - \frac{1}{9}\right] \\
&= 9\left[\left(x - \frac{1}{3}\right)^2 + \frac{16}{9}\right] \\
&= 9\left(x - \frac{1}{3}\right)^2 + 16 \\
&= 3^2\left(x - \frac{1}{3}\right)^2 + 16 \\
&= (3x-1)^2 + 16.
\end{aligned}$$

Now we let $u = 3x - 1$ so $du = 3\,dx$ and $\frac{1}{3}\,du = dx$. The integral becomes

$$\begin{aligned}
\int \frac{1}{9x^2 - 6x + 17}\,dx &= \frac{1}{(3x-1)^2 + 16}\,dx \\
&= \frac{1}{3}\int \frac{1}{u^2+16}\,du \\
&= \frac{1}{3}\cdot\frac{1}{4}\arctan\left(\frac{u}{4}\right) + C \\
&= \frac{1}{12}\arctan\left(\frac{3x-1}{4}\right) + C.
\end{aligned}$$

(d) Since

$$\int x^7 \sec^2(x^4)\,dx = \int x^4 \sec^2(x^4) \cdot x^3\,dx,$$

we let $u = x^4$ so $du = 4x^3\,dx$ and $\frac{1}{4}\,du = x^3\,dx$. The integral becomes

$$\int x^7 \sec^2(x^4)\,dx = \frac{1}{4}\int u \sec^2(u)\,du.$$

Now we use integration by parts with $w = u$ so $dw = du$, and $dv = \sec^2(u)\,du$ so $v = \tan(u)$. This yields

$$\begin{aligned}\int x^7 \sec^2(x^4)\,dx &= \frac{1}{4}\left[u\tan(u) - \int \tan(u)\,du\right]\\ &= \frac{1}{4}[u\tan(u) + \ln|\cos(u)|] + C\\ &= \frac{1}{4}x^4\tan(x^4) + \frac{1}{4}\ln|\cos(x^4)| + C.\end{aligned}$$

(e) First we write

$$\int \frac{4-x}{\sqrt{4-x^2}}\,dx = 4\int \frac{1}{\sqrt{4-x^2}}\,dx - \int \frac{x}{\sqrt{4-x^2}}\,dx.$$

The first integral is an elementary arcsine integral:

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin\left(\frac{x}{2}\right) + C.$$

For the second integral, we let $u = 4 - x^2$ so $du = -2x\,dx$ and $-\frac{1}{2}\,du = x\,dx$. Then

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = -\frac{1}{2}\left[2\sqrt{u}\right] + C = -\sqrt{4-x^2} + C.$$

Hence we have

$$\begin{aligned}\int \frac{4-x}{\sqrt{4-x^2}}\,dx &= 4\arcsin\left(\frac{x}{2}\right) - \left(-\sqrt{4-x^2}\right) + C\\ &= 4\arcsin\left(\frac{x}{2}\right) + \sqrt{4-x^2} + C.\end{aligned}$$

(f) Using long division of polynomials, we have

$$\begin{array}{r} 3x^2 - 4x \phantom{{}+5} \\ 2x+3 \,\big)\overline{\,6x^3 + x^2 - 12x + 5} \\ \underline{6x^3 + 9x^2 \phantom{{}-12x+5}} \\ -8x^2 - 12x + 5 \\ \underline{-8x^2 - 12x \phantom{{}+5}} \\ 5 \end{array}$$

Thus we can write the integral as

$$\int \frac{6x^3 + x^2 - 12x + 5}{2x + 3}\,dx = \int \left(3x^2 - 4x + \frac{5}{2x + 3}\right)\,dx$$

$$= 3 \cdot \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + 5 \cdot \frac{1}{2}\ln|2x + 3| + C$$

$$= x^3 - 2x^2 + \frac{5}{2}\ln|2x + 3| + C.$$