MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Section 1.4 | **Math 1001 Worksheet** | Winter 2024

SOLUTIONS

1. (a) Let $w = x$ so $dw = dx$, and let $dv = \cos(x)\,dx$ so $v = \sin(x)$. Then

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C.$$

(b) Let $w = x^2$ so $dw = 2x\,dx$, and let $dv = \cos(x)\,dx$ so $v = \sin(x)$. Then

$$\int x^2\cos(x)\,dx = x^2\sin(x) - 2\int x\sin(x)\,dx.$$

Now let $w = x$ so $dw = dx$ and let $dv = \sin(x)\,dx$ so $v = -\cos(x)$. Then we obtain

$$\int x^2\cos(x)\,dx = x^2\sin(x) - 2\left[-x\cos(x) + \int \cos(x)\,dx\right]$$

$$= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C.$$

(c) Let $w = x$ so $dw = dx$, and let $dv = \tan(x)\sec(x)\,dx$ so $v = \sec(x)$. Then

$$\int x\tan(x)\sec(x)\,dx = x\sec(x) - \int \sec(x)\,dx = x\sec(x) - \ln|\sec(x) + \tan(x)| + C.$$

(d) Let $w = y^4$ so $dw = 4y^3\,dy$. Let $dv = y^3e^{y^4}\,dy$. To integrate this to get v we must make a substitution; let $u = y^4$ so $\frac{1}{4}\,du = y^3\,dy$ and thus

$$v = \int dv = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u = \frac{1}{4}e^{y^4}.$$

Then, returning to the original integral, we have

$$\int y^7e^{y^4}\,dy = \int y^4y^3e^{y^4}\,dy = \frac{1}{4}y^4e^{y^4} - \int y^3e^{y^4}\,dy.$$

Using the same u-substitution as before, this becomes

$$\int y^7e^{y^4}\,dy = \frac{1}{4}y^4e^{y^4} - \frac{1}{4}\int e^u\,du = \frac{1}{4}y^4e^{y^4} - \frac{1}{4}e^u + C = \frac{1}{4}y^4e^{y^4} - \frac{1}{4}e^{y^4} + C.$$

Alternatively, we can make the substitution right away: let $u = y^4$ so $\frac{1}{4}\,du = y^3\,dy$. Then

$$\int y^7e^{y^4}\,dy = \int y^4y^3e^{y^4}\,dy = \frac{1}{4}\int ue^u\,du.$$

Now we use parts, setting $w = u$ so $dw = du$ and $dv = e^u\,dw$ so $v = e^u$. Then

$$\int y^7e^{y^4}\,dy = \frac{1}{4}\left[ue^u - \int e^u\,du\right] = \frac{1}{4}ue^u - \frac{1}{4}e^u + C = \frac{1}{4}y^4e^{y^4} - \frac{1}{4}e^{y^4} + C,$$

as before.

(e) Let $w = e^{3x}$ so $dw = 3e^{3x}\,dx$, and let $dv = \sin(5x)\,dx$ so $v = -\frac{1}{5}\cos(5x)$. Then

$$\int e^{3x}\sin(5x)\,dx = -\frac{1}{5}e^{3x}\cos(5x) + \frac{3}{5}\int e^{3x}\cos(5x)\,dx.$$

Again let $w = e^{3x}$ so $dw = 3e^{3x}\,dx$, and now let $dv = \cos(5x)\,dx$ so $v = \frac{1}{5}\sin(5x)$. This gives

$$\begin{aligned}\int e^{3x}\sin(5x)\,dx &= -\frac{1}{5}e^{3x}\cos(5x) + \frac{3}{5}\left[\frac{1}{5}e^{3x}\sin(5x) - \frac{3}{5}\int e^{3x}\sin(5x)\,dx\right]\\ &= -\frac{1}{5}e^{3x}\cos(5x) + \frac{3}{25}e^{3x}\sin(5x) - \frac{9}{25}\int e^{3x}\sin(5x)\,dx\\ \frac{34}{25}\int e^{3x}\sin(5x)\,dx &= -\frac{1}{5}e^{3x}\cos(5x) + \frac{3}{25}e^{3x}\sin(5x)\\ \int e^{3x}\sin(5x)\,dx &= -\frac{5}{34}e^{3x}\cos(5x) + \frac{3}{34}e^{3x}\sin(5x) + C.\end{aligned}$$

(f) Let $w = \cos\left(\frac{2x}{3}\right)$ so $dw = -\frac{2}{3}\sin\left(\frac{2x}{3}\right)\,dx$, and let $dv = \cos(x)\,dx$ so $v = \sin(x)$. Then

$$\int \cos(x)\cos\left(\frac{2x}{3}\right)\,dx = \sin(x)\cos\left(\frac{2x}{3}\right) + \frac{2}{3}\int \sin(x)\sin\left(\frac{2x}{3}\right)\,dx.$$

Now let $w = \sin\left(\frac{2x}{3}\right)\,dx$ so $dw = \frac{2}{3}\cos\left(\frac{2x}{3}\right)\,dx$, and let $dv = \sin(x)\,dx$ so $v = -\cos(x)\,dx$. We now obtain

$$\begin{aligned}\int \cos(x)\cos\left(\frac{2x}{3}\right)\,dx &= \sin(x)\cos\left(\frac{2x}{3}\right)\\ &\quad + \frac{2}{3}\left[-\cos(x)\sin\left(\frac{2x}{3}\right) + \frac{2}{3}\int \cos(x)\cos\left(\frac{2x}{3}\right)\,dx\right]\\ &= \sin(x)\cos\left(\frac{2x}{3}\right) - \frac{2}{3}\cos(x)\sin\left(\frac{2x}{3}\right)\\ &\quad + \frac{4}{9}\int \cos(x)\cos\left(\frac{2x}{3}\right)\,dx\\ \frac{5}{9}\int \cos(x)\cos\left(\frac{2x}{3}\right)\,dx &= \sin(x)\cos\left(\frac{2x}{3}\right) - \frac{2}{3}\cos(x)\sin\left(\frac{2x}{3}\right)\\ \int \cos(x)\cos\left(\frac{2x}{3}\right)\,dx &= \frac{9}{5}\sin(x)\cos\left(\frac{2x}{3}\right) - \frac{6}{5}\cos(x)\sin\left(\frac{2x}{3}\right) + C.\end{aligned}$$

(g) Let $w = \arcsin(6x)$ so $dw = \dfrac{6}{\sqrt{1-36x^2}}$. Let $dv = dx$ so $v = x$. Then

$$\int \arcsin(6x)\,dx = x\arcsin(6x) - 6\int \frac{x}{\sqrt{1-36x^2}}\,dx.$$

Now we use u-substitution with $u = 1 - 36x^2$ so $-\frac{1}{72}\,du = x\,dx$. The integral becomes

$$\begin{aligned}\int \arcsin(6x)\,dx &= x\arcsin(6x) + 6\left[\frac{1}{72}\int \frac{1}{\sqrt{u}}\,du\right] \\ &= x\arcsin(6x) + \frac{1}{12}\left[2\sqrt{u}\right] + C \\ &= x\arcsin(6x) + \frac{1}{6}\sqrt{1-36x^2} + C.\end{aligned}$$

2. (a) We use u-substitution. Let $u = x^2 - 9$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. The integral becomes

$$\begin{aligned}\int \frac{x}{\sqrt{x^2-9}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\ &= \frac{1}{2}\int u^{-\frac{1}{2}}\,du \\ &= \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= \sqrt{x^2-9} + C.\end{aligned}$$

(b) This is an arcsecant type integral with $k = 3$, so

$$\int \frac{1}{x\sqrt{x^2-9}}\,dx = \int \frac{1}{x\sqrt{x^2-3^2}}\,dx = \frac{1}{3}\operatorname{arcsec}\left(\frac{x}{3}\right) + C.$$

(c) We use integration by parts. Let $w = x$ so $dw = dx$, and let $dv = \csc^2(9x)\,dx$ so $v = -\frac{1}{9}\cot(9x)$. Then

$$\begin{aligned}\int x\csc^2(9x)\,dx &= -\frac{1}{9}x\cot(9x) + \frac{1}{9}\int \cot(9x)\,dx \\ &= -\frac{1}{9}x\cot(9x) + \frac{1}{9}\cdot\frac{1}{9}\ln|\sin(9x)| + C \\ &= -\frac{1}{9}x\cot(9x) + \frac{1}{81}\ln|\sin(9x)| + C.\end{aligned}$$

(d) We use u-substitution. Let $u = x^5$ so $du = 5x^4\,dx$ and $\frac{1}{5}\,du = x^4\,dx$. The integral becomes

$$\begin{aligned}\int x^4 e^{x^5}\,dx &= \frac{1}{5}\int e^u\,du \\ &= \frac{1}{5}e^u + C \\ &= \frac{1}{5}e^{x^5} + C.\end{aligned}$$

(e) We begin with u-substitution. Let $u = x^5$ so $du = 5x^4\,dx$ and $\frac{1}{5}\,du = x^4\,dx$. The integral becomes

$$\int x^9 e^{x^5}\,dx = \int x^5 e^{x^5}(x^4\,dx) = \frac{1}{5}\int ue^u\,du.$$

Now we use integration by parts, letting $w = u$ so $dw = du$, and $dv = e^u\,du$ so $v = e^u$. We obtain

$$\begin{aligned}\int x^9 e^{x^5}\,dx &= \frac{1}{5}\left[ue^u - \int e^u\,du\right] \\ &= \frac{1}{5}ue^u - \frac{1}{5}e^u + C \\ &= \frac{1}{5}x^5 e^{x^5} - \frac{1}{5}e^{x^5} + C.\end{aligned}$$

(f) We begin by completing the square:

$$\begin{aligned}9x^2 - 12x + 8 &= 9\left[\left(x^2 - \frac{4}{3}x\right) + \frac{8}{9}\right] \\ &= 9\left[\left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) + \frac{8}{9} - \frac{4}{9}\right] \\ &= 9\left[\left(x - \frac{2}{3}\right)^2 + \frac{4}{9}\right] \\ &= (3x-2)^2 + 4.\end{aligned}$$

Thus the integral becomes

$$\int \frac{1}{9x^2 - 12x + 8}\,dx = \frac{1}{(3x-2)^2 + 4}\,dx.$$

Now let $u = 3x - 2$ so $du = 3\,dx$ and $\frac{1}{3}\,du = dx$. Finally,

$$\begin{aligned}\int \frac{1}{9x^2 - 12x + 8}\,dx &= \frac{1}{3}\int \frac{1}{u^2+4}\,du \\ &= \frac{1}{3}\cdot\frac{1}{2}\arctan\left(\frac{u}{2}\right) + C \\ &= \frac{1}{6}\arctan\left(\frac{3x-2}{2}\right) + C.\end{aligned}$$

(g) We try integration by parts, with $w = e^{4x}$ so $dw = 4e^{4x}\,dx$ and $dv = \cos(x)\,dx$ so $v = \sin(x)$. Then

$$\int e^{4x}\cos(x)\,dx = e^{4x}\sin(x) - 4\int e^{4x}\sin(x)\,dx.$$

We try integration by parts a second time. Again, we let $w = e^{4x}$ so $dw = 4e^{4x}\,dx$, and now we let $dv = \sin(x)\,dx$ so $v = -\cos(x)$. Thus

$$\begin{aligned}
\int e^{4x}\cos(x)\,dx &= e^{4x}\sin(x) - 4\left[-e^{4x}\cos(x) + 4\int e^{4x}\cos(x)\,dx\right] \\
&= e^{4x}\sin(x) + 4e^{4x}\cos(x) - 16\int e^{4x}\cos(x)\,dx \\
17\int e^{4x}\cos(x)\,dx &= e^{4x}\sin(x) + 4e^{4x}\cos(x) + C \\
\int e^{4x}\cos(x)\,dx &= \frac{1}{17}e^{4x}\sin(x) + \frac{4}{17}e^{4x}\cos(x) + C.
\end{aligned}$$

(h) We begin by performing long division:

$$\begin{array}{r}
6x - 1 \\
2x-5 \,\big)\, 12x^2 - 32x + 14 \\
12x^2 - 30x \quad\;\; \\
\hline
-2x + 14 \\
-2x + \;\;5 \\
\hline
9
\end{array}$$

Now we can write

$$\begin{aligned}
\int \frac{12x^2 - 32x + 14}{2x-5}\,dx &= \int \left(6x - 1 + \frac{9}{2x-5}\right)\,dx \\
&= 3x^2 - x + \frac{9}{2}\ln|2x-5| + C.
\end{aligned}$$

(i) Let $u = \ln(x)$ so $du = \frac{1}{x}\,dx$. The integral becomes

$$\begin{aligned}
\int \frac{1}{x\sqrt{4 - \ln^2(x)}}\,dx &= \int \frac{1}{\sqrt{4-u^2}}\,du \\
&= \arcsin\left(\frac{u}{2}\right) + C \\
&= \arcsin\left(\frac{\ln(x)}{2}\right) + C.
\end{aligned}$$

(j) Recall that $1 + \tan^2(x) = \sec^2(x)$, so

$$\int \cos^2(x)[1 + \tan^2(x)]\,dx = \cos^2(x)\sec^2(x)\,dx = \int dx = x + C.$$

Alternatively, we could write

$$\int \cos^2(x)[1 + \tan^2(x)]\,dx = \int [\cos^2(x) + \sin^2(x)]\,dx = \int dx = x + C.$$

3. (a) Let $w = \sin^{n-1}(x)$ so $dw = (n-1)\sin^{n-2}(x)\cos(x)\,dx$. Let $dv = \sin(x)\,dx$ so $v = -\cos(x)$. Then

$$\begin{aligned}
\int \sin^n(x)\,dx &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx \\
&= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)[1-\sin^2(x)]\,dx \\
&= -\cos(x)\sin^{n-1}(x) \\
&\quad + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx \\
n\int \sin^n(x)\,dx &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx \\
\int \sin^n(x)\,dx &= -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.
\end{aligned}$$

(b) Using the reduction formula with $n = 7$, we obtain

$$\int \sin^7(x)\,dx = -\frac{1}{7}\cos(x)\sin^6(x) + \frac{6}{7}\int \sin^5(x)\,dx.$$

Now using the formula with $n = 5$, this becomes

$$\begin{aligned}
\int \sin^7(x)\,dx &= -\frac{1}{7}\cos(x)\sin^6(x) + \frac{6}{7}\left[-\frac{1}{5}\cos(x)\sin^4(x) + \frac{4}{5}\int \sin^3(x)\,dx\right] \\
&= -\frac{1}{7}\cos(x)\sin^6(x) - \frac{6}{35}\cos(x)\sin^4(x) + \frac{24}{35}\int \sin^3(x)\,dx.
\end{aligned}$$

We use the formula once more, this time with $n = 3$, and get

$$\begin{aligned}
&\int \sin^7(x)\,dx \\
&= -\frac{1}{7}\cos(x)\sin^6(x) - \frac{6}{35}\cos(x)\sin^4(x) \\
&\quad + \frac{24}{35}\left[-\frac{1}{3}\cos(x)\sin^2(x) + \frac{2}{3}\int \sin(x)\,dx\right] \\
&= -\frac{1}{7}\cos(x)\sin^6(x) - \frac{6}{35}\cos(x)\sin^4(x) - \frac{8}{35}\cos(x)\sin^2(x) - \frac{16}{35}\cos(x) + C.
\end{aligned}$$