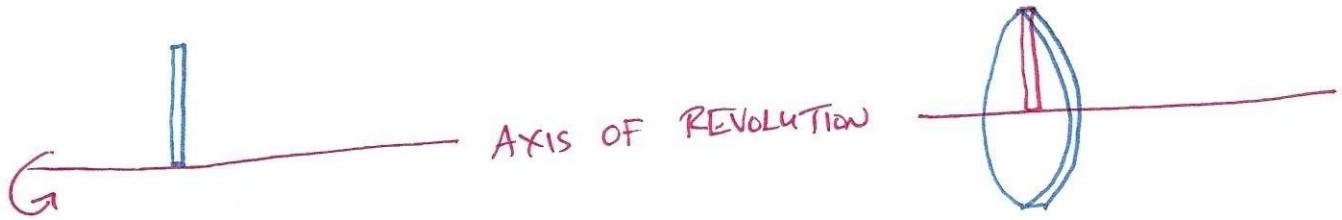


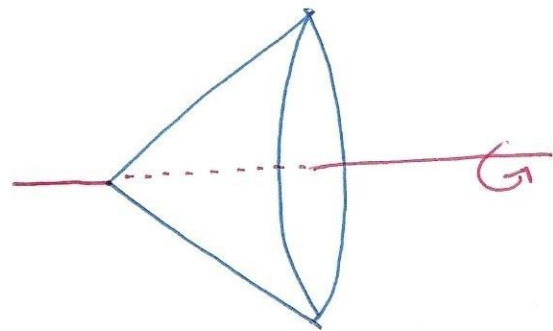
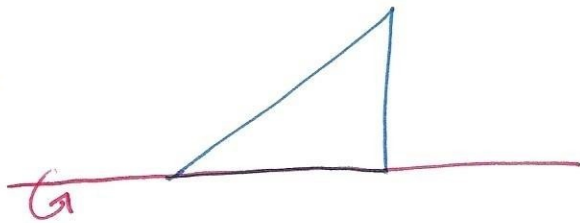
Section 4.6: Volumes

Consider a plane shape, such as a rectangle. Suppose we have a line which does not pass through the plane shape, called the axis of revolution. Then we can revolve the plane shape around the axis of revolution to obtain a symmetric three-dimensional shape, called a solid of revolution.



More complicated plane shapes generate more complicated solids of revolution.

eg



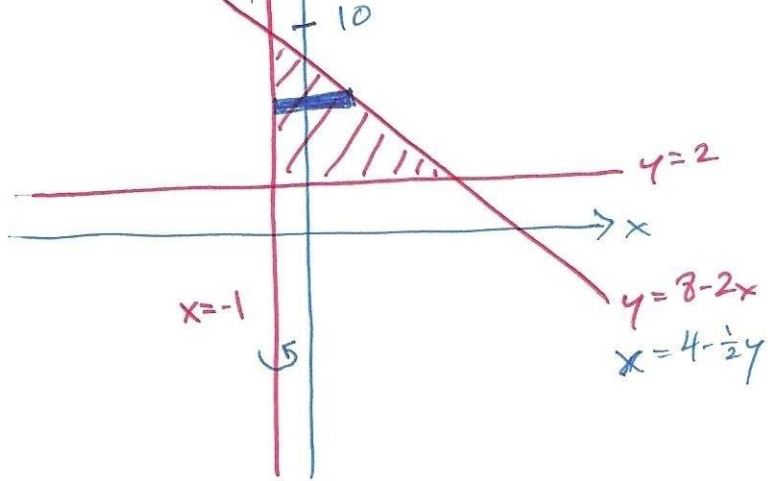
Our goal is to find the volume of a solid of revolution.

Let's assume that we have a plane shape which lies under a curve $y=f(x)$ and above the x -axis on $[a,b]$ where the x -axis is also the axis of revolution. Then we can divide the plane shape into n rectangles, each of width Δx and height $f(x_i^*)$ where x_i^* is a sample point on each of the resulting subintervals.

When we rotate each rectangle around the axis of revolution, we obtain a disc of width Δx and radius $f(x_i^*)$.

Then the volume of each disc is $\pi [f(x_i^*)]^2 \Delta x$ and so the volume V of the solid of revolution is given by

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi [f(x_i^*)]^2 \Delta x$$
$$= \pi \lim_{n \rightarrow \infty} \sum_{i=1}^n [f(x_i^*)]^2 \Delta x$$



$$2x = 8 - y$$

$$x = 4 - \frac{1}{2}y$$

We set

$$4 - \frac{1}{2}y = -1$$

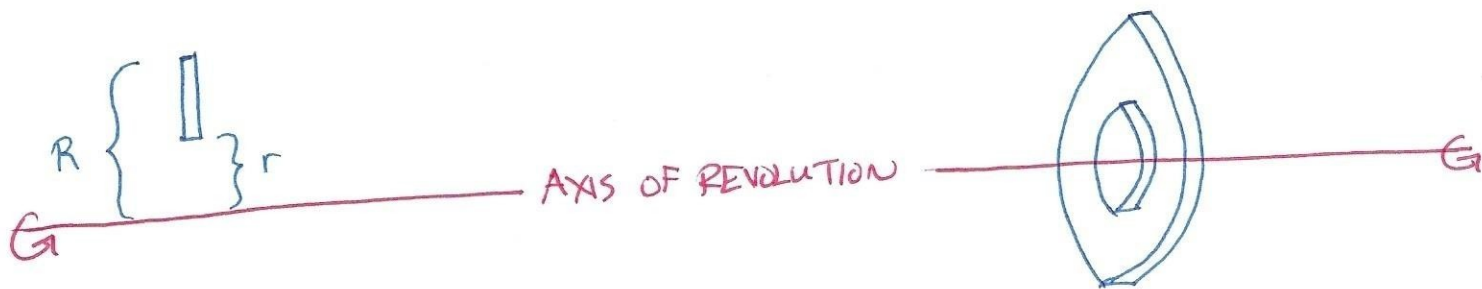
$$5 = \frac{1}{2}y \rightarrow y = 10$$

By the Disc Method,

$$V = \pi \int_2^{10} \left[\left(4 - \frac{1}{2}y\right) - (-1) \right]^2 dy$$

$$= \pi \int_2^{10} \left(5 - \frac{1}{2}y\right)^2 dy \quad \boxed{= \frac{128\pi}{3}}$$

Now suppose that the axis of revolution is not a boundary curve of the plane shape.



In this situation, the solid generated by a rectangle is a washer, which is a large disc with a smaller disc removed from its centre.

This is the disc method.

Let R be the distance from the axis of revolution to the farther side of the rectangle. We call R the outer radius of the washer.

Let r be the distance from the axis of revolution to the closer side of the rectangle. We call r the inner radius of the washer.

Then the volume of the washer is given by

$$V = \pi R^2 - \pi r^2 = \pi (R^2 - r^2)$$

and hence the volume of the solid of revolution is

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx$$

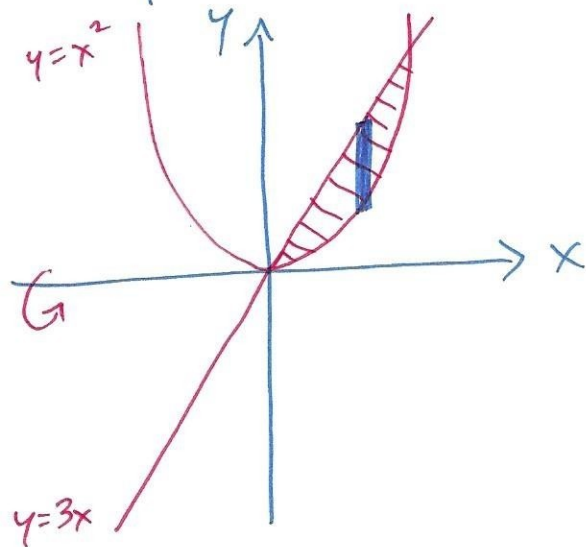
In the case where the bottom boundary curve is the axis of revolution, $r(x) \equiv 0$ and so we have

$$V = \pi \int_a^b ([R(x)]^2 - 0^2) dx = \pi \int_a^b [R(x)]^2 dx$$

Thus we have now developed the Disc-Washer Method for volume.

eg Find the volume of the solid obtained by revolving around the x-axis the region bounded by $y=3x$

and $y=x^2$.



Here, $R(x) = 3x$

$r(x) = x^2$

We set $x^2 = 3x$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x=0 \text{ and } x=3$$

$$V = \pi \int_0^3 ([3x]^2 - [x^2]^2) dx$$

$$= \pi \int_0^3 (9x^2 - x^4) dx$$

$$= \frac{162\pi}{5}$$

———— THE END ————