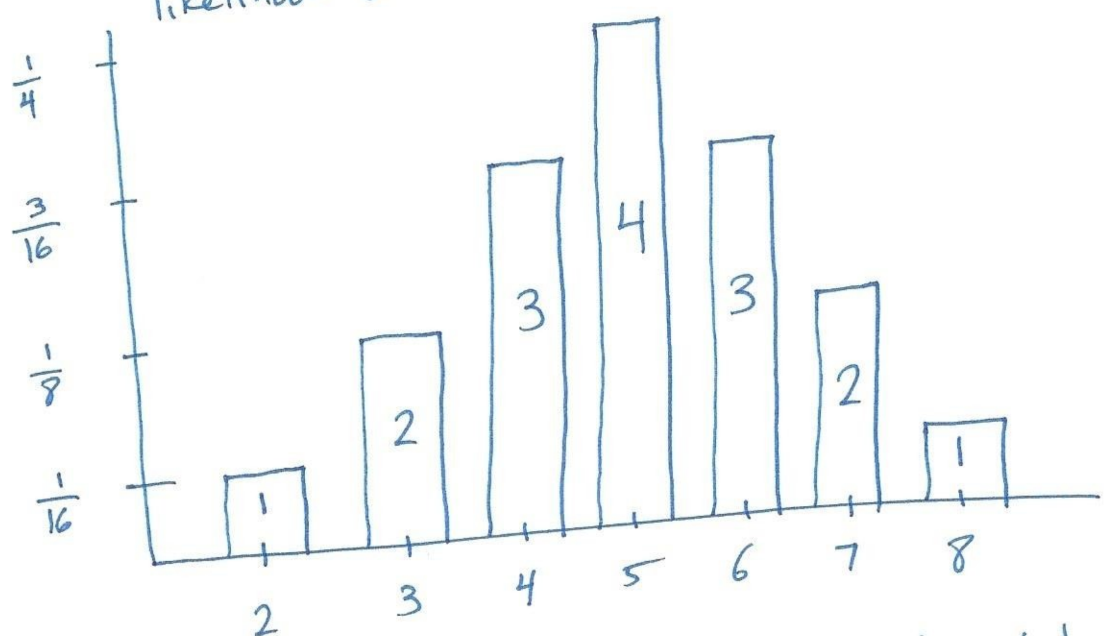
Section 4.5: Probability

Elementary probability typically concerns itself with the study of discrete random variables, in which the possible outcomes of a scenario are discontinuous.

eg Suppose we roll two 4-sided dice and add together the results. The possible outcomes are 2, 3, 4, 5, 6, 7 or 8. We can depict the likelihood of each outcome as follows:



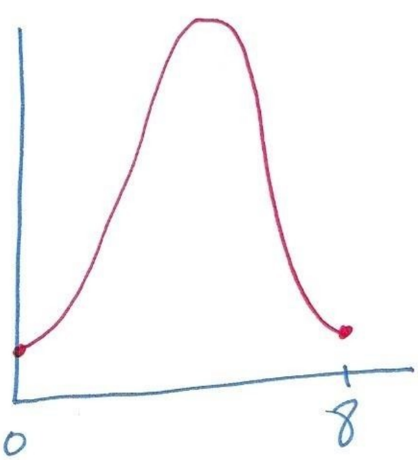
We normally scale probabilities such that the total probability of all possible outcomes is exactly 1.

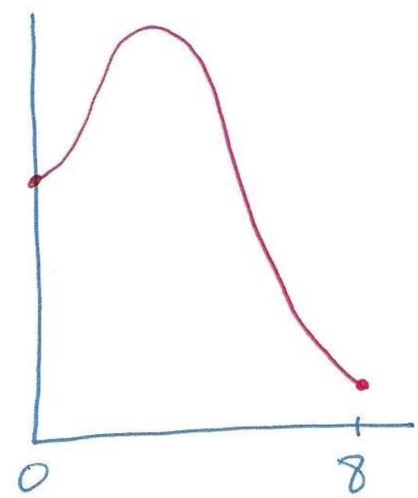
For discrete random variables, we can consider the probability of a single outcome, or of multiple outcomes, such as all the outcomes which lie on a given interval.

When studying continuous random variables, the possible outcomes will be defined by a continuous function.

eg Suppose you choose two videos from your YouTube search history and add together your viewing time from the first 4 minutes of each. The possible outcomes include any real number between 0 and 8.

Depending on one's viewing habits, many different functions might depict the relative likelihood of the possible outcomes such as:





The function that describes the possible outcomes of a scenario is a probability density function $f(x)$.

For continuous random variables, we are only interested in the probability of outcomes which lie on the interval $[a, b]$, denoted by $P(a \leq X \leq b)$.

Given a probability density function $f(x)$,

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

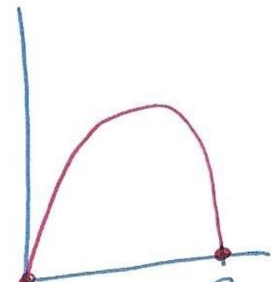
Note that a probability density function must obey the following properties:

① $f(x) \geq 0$ for all real numbers x

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

eg Suppose we want to model our YouTube search history viewing habits with a parabolic function. Then

$$f(x) = \begin{cases} kx(8-x), & \text{for } 0 \leq x \leq 8 \\ 0, & \text{otherwise} \end{cases}$$



Determine the value of the constant k which ensures that $f(x)$ is a probability density function.

First, we note that $f(x)$ is non-negative as long as $k \geq 0$.

Next, we have $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^8 kx(8-x)\,dx$

$$= k\int_0^8 (8x - x^2)\,dx$$

$$= k\left[8 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_0^8$$

$$= k\left[4x^2 - \frac{1}{3}x^3\right]_0^8$$

$$= \frac{256}{3}k$$

We must have $\frac{256}{3}k = 1$, so $\boxed{k = \frac{3}{256}}$.

eg Determine the probability that the total viewing time of the two videos is 6 minutes or longer.

$$P(X \geq 6) = P(6 \leq X \leq 8)$$
$$= \int_6^8 f(x)\,dx$$
$$= \int_6^8 \frac{3}{256} x(8-x)\,dx$$
$$= \frac{3}{256} \int_6^8 (8x - x^2)\,dx$$
$$= \frac{3}{256} \left[4x^2 - \frac{x^3}{3} \right]_6^8 \boxed{= \frac{5}{32}}$$

In general, the probability of an outcome falling in the interval $[x_{i-1}, x_i]$ is approximately $f(x_i^*)\Delta x_i$ where x_i^* is a sample point on $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$.

Let N be the total number of events under consideration. Then the number of events falling into $[x_{i-1}, x_i]$ is given by $N f(x_i^*)\Delta x_i$.

The total viewing time of these events is about

$$x_i^* N f(x_i^*) \Delta x_i$$

and the total viewing time of all the events is

$$\sum_{i=1}^{n} x_i^* N f(x_i^*) \Delta x_i = N \sum_{i=1}^{n} x_i^* f(x_i^*) \Delta x_i .$$

The average viewing time is given

$$\frac{N \sum_{i=1}^{n} x_i^* f(x_i^*) \Delta x_i}{N} = \sum_{i=1}^{n} x_i^* f(x_i^*) \Delta x_i .$$

This is a Riemann sum corresponding to the definite integral

$$\mu = \int_a^b x f(x) dx$$

eg Determine the mean total viewing time of two videos, using the probability density function previously established.

$$\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_0^8 x \cdot \frac{3}{256} x (8-x) dx$$

$$= \frac{3}{256} \int_0^8 (8x^2 - x^3) dx$$

$$= \frac{3}{256} \left[\frac{8}{3} x^3 - \frac{1}{4} x^4 \right]_0^8$$

$$\boxed{= 4}$$