Section 4.1: Modelling with Differential Equations

In real-world applications, we often have information about the rate of change of a process, and our goal is to determine information about the function which describes the process itself.

eg A simple way to model the growth of a population is to assume that its rate of change is proportional to its current size. If $y(t)$ represents the size of the population at time t, we could write

$$\frac{dy}{dt} = ky$$

for some constant k.

A <u>differential equation</u> (DE) is an equation which involves an unknown function and one or more of its derivatives.

We want to identify functions which satisfy the DE, so that the two sides of the DE are the same when y is substituted into the equation.

eg Given the DE $\frac{dy}{dt} = ky$, determine whether $y(t) = 5e^{kt}$ and $y(t) = 5\sin(kt)$ are solutions of the DE.

For $y(t) = 5e^{kt}$, we have

$$\frac{dy}{dt} = \frac{d}{dt}\left[5e^{kt}\right] = 5 \cdot e^{kt} \cdot k = 5ke^{kt}$$

$$ky = k \cdot 5e^{kt} = 5ke^{kt}$$

Since these are equal, $y(t) = 5e^{kt}$ <u>is</u> a solution of the DE.

For $y(t) = 5\sin(kt)$ we have

$$\frac{dy}{dt} = \frac{d}{dt}\left[5\sin(kt)\right] = 5\cos(kt) \cdot k = 5k\cos(kt)$$

$$ky = k \cdot 5\sin(kt) = 5k\sin(kt)$$

These are not equal, so $y(t) = 5\sin(kt)$ is <u>not</u> a solution of the DE.

Typically, a DE will be satisfied by a family of functions called its <u>general solution</u>. It is often found through integration.

The simplest type of DE is one in which the unknown function appears only in the form of its derivative.

eg Consider the DE $\frac{dy}{dt} - 2t^3 = 3$.

We can rewrite the DE as

$$\frac{dy}{dt} = 2t^3 + 3$$

so $y(t) = \int \frac{dy}{dt}\,dt = \int (2t^3+3)\,dt$

$$= 2 \cdot \frac{t^4}{4} + 3t + C$$

$$= \frac{1}{2}t^4 + 3t + C$$

The general solution is $\boxed{y(t) = \frac{1}{2}t^4 + 3t + C}$.

In practice, we often have an additional piece of information about the unknown function, called an initial condition. A specific function which satisfies both the DE and the initial condition is called the particular solution.

The combination of a DE and an initial condition is called an initial value problem (IVP).

eg Find the particular solution of the IVP

$$t^2 \frac{dy}{dt} - 3t^4 + 1 = 0, \quad y(1) = 0.$$

First we find the general solution:

$$t^2 \frac{dy}{dt} = 3t^4 - 1$$

$$\frac{dy}{dt} = 3t^2 - t^{-2}$$

$$y(t) = \int (3t^2 - t^{-2})\,dt$$

$$= 3 \cdot \frac{t^3}{3} - \frac{t^{-1}}{-1} + C$$

$$= t^3 + \frac{1}{t} + C$$

We are given that $y=0$ when $t=1$, so we substitute into the general solution to get

$$0 = 1^3 + \frac{1}{1} + C$$

$$0 = 2 + C \rightarrow C = -2$$

Therefore the particular solution is

$$\boxed{y(t) = t^3 + \frac{1}{t} - 2}.$$

We often apply DEs to the study of objects in motion or <u>kinematics</u>. We assume that an object moves such that its position is given by a function $s(t)$, its velocity is $v(t)$, and its acceleration is $a(t)$. We know that

$$s'(t) = v(t) \quad \text{and} \quad v'(t) = a(t).$$

Thus we can also write

$$s(t) = \int v(t)\,dt \quad \text{and} \quad v(t) = \int a(t)\,dt.$$

eg A ball is thrown upwards by a girl standing on a cliff that is 50 m high. She throws it with a velocity of 28 m/sec. To two decimal places, determine the time at which the ball reaches its maximum height, and measure that height.

We assume that gravity is the only force acting on the ball, so $a(t) = -9.8$. We are given that $s(0) = 50$ and $v(0) = 28$.

Then
$$\begin{aligned} v(t) &= \int (-9.8)\,dt \\ &= -9.8t + C \end{aligned}$$

Since $v(0) = 28$, we have
$$\begin{aligned} 28 &= -9.8 \cdot 0 + C \\ 28 &= C \end{aligned}$$

and hence the particular solution is $v(t) = -9.8t + 28$.

Next,

$$s(t) = \int (-9.8t + 28)\, dt$$

$$= -9.8 \cdot \frac{t^2}{2} + 28t + D$$

$$= -4.9t^2 + 28t + D$$

Since $s(0) = 50$, we obtain $50 = -4.9 \cdot 0 + 28 \cdot 0 + D$

$$50 = D$$

Thus the particular solution is $s(t) = -4.9t^2 + 28t + 50$.

Now we set $v(t) = 0$ so

$$-9.8t + 28 = 0$$

$$t = \frac{28}{9.8} \approx 2.86$$

The ball reaches its maximum height at about 2.86 sec.

Finally, $s(2.86) = -4.9 \cdot (2.86)^2 + 28 \cdot (2.86) + 50$

$$= 90$$

so the maximum height achieved by the ball is 90 m.

DEs may involve derivatives of the unknown function of any order. For instance, we can write

$$\frac{d^2 s}{dt^2} = a(t)$$

This is a second-order DE.

The <u>order</u> of a DE is the order of the highest derivative which appears in the equation.

eg $\frac{dy}{dt} = ky$ is a <u>first-order</u> DE

eg Solve the IVP

$$f''(t) - \cos\left(\tfrac{1}{2}t\right) = 0$$

given initial conditions $f(0) = 0$ and $f(\pi) = 0$.

We have

$$f''(t) = \cos\left(\tfrac{1}{2}t\right)$$

$$f'(t) = \int \cos\left(\tfrac{1}{2}t\right) dt$$

$$= \frac{\sin(\frac{1}{2}t)}{\frac{1}{2}} + C_1$$

$$= 2\sin\left(\tfrac{1}{2}t\right) + C_1$$

$$f(t) = \int \left(2\sin\left(\tfrac{1}{2}t\right) + C_1\right) dt$$

$$= 2 \cdot \frac{-\cos(\frac{1}{2}t)}{\frac{1}{2}} + C_1 t + C_2$$

$$= -4\cos\left(\tfrac{1}{2}t\right) + C_1 t + C_2$$

Since $f(0) = 0$, we have

$$0 = -4\cos(0) + C_1 \cdot 0 + C_2$$

$$0 = -4 + C_2$$

$$C_2 = 4$$

Now we have

$$f(t) = -4\cos\left(\tfrac{1}{2}t\right) + C_1 t + 4$$

Since $f(\pi) = 0$, we have

$$0 = -4\cos\left(\tfrac{\pi}{2}\right) + C_1 \cdot \pi + 4$$

$$0 = \pi C_1 + 4$$

$$C_1 = -\frac{4}{\pi}$$

The particular solution is $\boxed{f(t) = -4\cos\left(\tfrac{1}{2}t\right) - \tfrac{4}{\pi}t + 4}$.

Higher-order DEs also appear in many applications.

eg Hooke's Law states that the force exerted by a spring to return to its equilibrium state is proportional to its displacement $s(t)$. Since force is the product of mass and acceleration, Hooke's Law can be expressed as the second-order DE

$$m\frac{d^2s}{dt^2} = ks$$

for some constant k.