Section 3.4: Improper Integrals

Given a definite integral $\int_a^b f(x)\,dx$, we say that it is improper if it does not fulfill the requirements of FTC ②. There are two ways this can happen:

① The interval of integration could be infinite or semi-infinite. That is, b could be ∞, a could be $-\infty$, or both.

② The integrand $f(x)$ could be discontinuous on the interval of integration.

We want to determine whether it's possible to evaluate each kind of improper integral and, if so, how.

Case ①: First, suppose we have a definite integral of the form $\int_a^\infty f(x)\,dx$. We define

$$\int_a^\infty f(x)\,dx = \lim_{T\to\infty} \int_a^T f(x)\,dx$$

so that FTC ② can be applied to the definite integral involving the dummy variable T, since it can be taken to represent an arbitrary real number. If the limit exists, we say that the improper integral is convergent. Otherwise, it is divergent.

eg $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{T\to\infty} \int_1^T \frac{1}{x^2}\,dx$

$= \lim_{T\to\infty} \left[-\frac{1}{x}\right]_1^T$

$= \lim_{T\to\infty} \left[-\frac{1}{T} + 1\right] = 0 + 1 \boxed{= 1}$

This improper integral is <u>convergent</u>.

eg $\int_1^\infty \frac{1}{x}\,dx = \lim_{T\to\infty} \int_1^T \frac{1}{x}\,dx$

$= \lim_{T\to\infty} \left[\ln|x|\right]_1^T$

$= \lim_{T\to\infty} \left[\ln|T| - 0\right] = \infty$

This improper integral is <u>divergent</u>.

eg $\int_0^\infty \cos(x)\,dx = \lim_{T\to\infty} \int_0^T \cos(x)\,dx$

$= \lim_{T\to\infty} \left[\sin(x)\right]_0^T$

$= \lim_{T\to\infty} \left[\sin(T) - 0\right]$ which does <u>not</u> exist

This improper integral <u>diverges</u>.

Now suppose that we have an integral of the form $\int_{-\infty}^{b} f(x)dx$.

Then we define

$$\int_{-\infty}^{b} f(x)dx = \lim_{T \to -\infty} \int_{T}^{b} f(x)\,dx.$$

eg
$$\int_{-\infty}^{1} e^{2x}\,dx = \lim_{T \to -\infty} \int_{T}^{1} e^{2x}\,dx$$
$$= \lim_{T \to -\infty} \left[\frac{1}{2}e^{2x}\right]_{T}^{1}$$
$$= \lim_{T \to -\infty} \left[\frac{1}{2}e^{2} - \frac{1}{2}e^{2T}\right]$$
$$= \frac{1}{2}e^{2} - 0 \; \boxed{= \frac{1}{2}e^{2}} \quad \text{(convergent)}$$

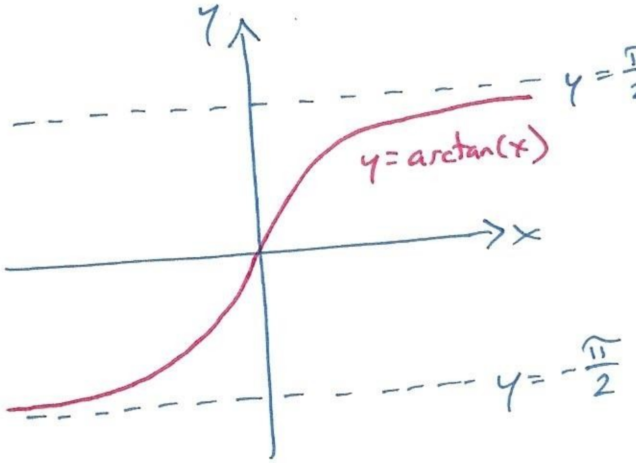
Finally, suppose we have an integral of the form $\int_{-\infty}^{\infty} f(x)\,dx$.
We choose an appropriate value p and use the Additive Interval Property to write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{p} f(x)dx + \int_{p}^{\infty} f(x)dx.$$

eg
$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

First we have

$$\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx = \lim_{T\to-\infty} \int_{T}^{0} \frac{1}{x^2+1}\,dx$$



$$= \lim_{T\to-\infty} \Big[\arctan(x)\Big]_{T}^{0}$$

$$= \lim_{T\to-\infty} \Big[0 - \arctan(T)\Big]$$

$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Next we have

$$\int_{0}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{T\to\infty} \int_{0}^{T} \frac{1}{x^2+1}\,dx$$

$$= \lim_{T\to\infty} \Big[\arctan(x)\Big]_{0}^{T}$$

$$= \lim_{T\to\infty} \Big[\arctan(T) - 0\Big]$$

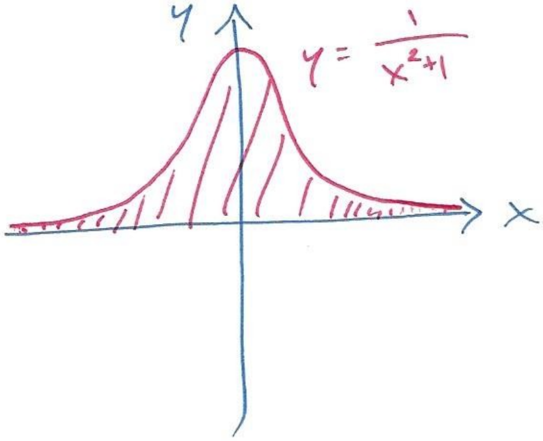
$$= \frac{\pi}{2}$$

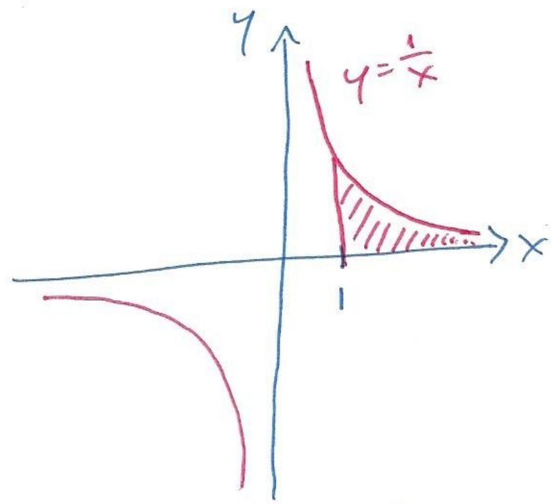
Thus $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2} + \frac{\pi}{2} \boxed{= \pi}$ (convergent)

As long as $f(x) \geq 0$, we can still interpret $\int_a^b f(x)\,dx$ as representing the area under $y = f(x)$ on $[a, b]$ even when the integral is improper.

Some regions of infinite extent can have a finite area.

eg The region under $y = \frac{1}{x^2+1}$ on $(-\infty, \infty)$ has $A = \pi$.





Other regions of infinite extent can have an infinite area,

eg We showed that $\int_1^\infty \frac{1}{x}\,dx$ is divergent so no finite value can be assigned to the area under $y = \frac{1}{x}$ on the interval $[1, \infty)$.

We can use any appropriate integration technique to evaluate an improper integral.

eg $\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx = \lim_{T\to\infty} \int_0^T \frac{2x}{(x^2+1)^2}\,dx$

We let $u = x^2+1$ so $du = 2x\,dx$

When $x=0$, $u=1$

$x=T$, $u=T^2+1$

The integral becomes

$$\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx = \lim_{T\to\infty} \int_1^{T^2+1} u^{-2}\,du$$

$$= \lim_{T\to\infty} \left[\frac{u^{-1}}{-1}\right]_1^{T^2+1}$$

$$= \lim_{T\to\infty} \left[\frac{-1}{T^2+1} + 1\right]$$

$$= 0+1 \boxed{=1} \quad \text{(convergent)}$$

For the second type of improper integral, let's first consider the case where we have $\int_a^b f(x)\,dx$ where $f(x)$ is discontinuous at the lower bound $x=a$. We write

$$\int_a^b f(x)\,dx = \lim_{T\to a^+} \int_T^b f(x)\,dx$$

eg $\int_0^4 \frac{1}{\sqrt{x}}\,dx$

Note that $\frac{1}{\sqrt{x}}$ is discontinuous at $x=0$, so we write

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{T\to 0^+} \int_T^4 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{T\to 0^+} \left[\frac{x^{1/2}}{1/2}\right]_T^4$$

$$= 2 \lim_{T\to 0^+} \left[2 - T^{1/2}\right]$$

$$= 2\cdot(2-0) \boxed{=4} \quad \text{(convergent)}$$

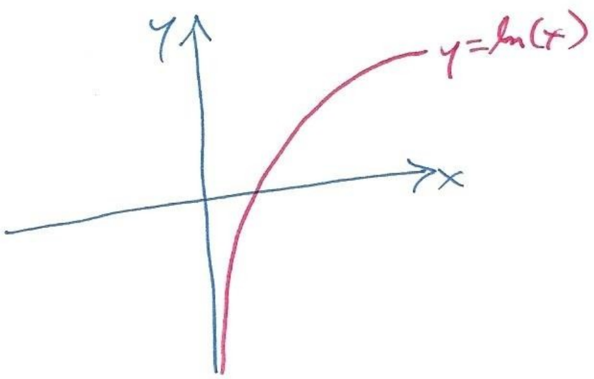
If $f(x)$ is discontinuous at the upper bound $x=b$, we write

$$\int_a^b f(x)\,dx = \lim_{T\to b^-} \int_a^T f(x)\,dx.$$

eg $\int_{-2}^{2} \frac{1}{x-2}\,dx$

Note that $\frac{1}{x-2}$ is discontinuous at $x=2$, so we have

$$\int_{-2}^{2} \frac{1}{x-2}\,dx = \lim_{T\to 2^-} \int_{-2}^{T} \frac{1}{x-2}\,dx$$



$$= \lim_{T\to 2^-} \Big[\ln|x-2|\Big]_{-2}^{T}$$

$$= \lim_{T\to 2^-} \Big[\ln|T-2| - \ln(4)\Big]$$

$$= -\infty$$

This improper integral is <u>divergent</u>.

If $f(x)$ is discontinuous at $x=p$ where $a<p<b$ then we apply the Additive Interval Property to write

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$$

eg $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

We note that $\dfrac{1}{\sqrt[3]{x-1}}$ is discontinuous at $x=1$, so we write

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$= -\frac{3}{2} + 6 \ \boxed{= \frac{9}{2}} \quad \text{(convergent)}$$

We often need to use l'Hôpital's Rule to evaluate the limit associated with an improper integral.

eg $\displaystyle\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx$

Since $x=0$ is a discontinuity of $\dfrac{\ln(x)}{\sqrt{x}}$, we write

$$\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx = \lim_{T\to 0^+} \int_T^1 \frac{\ln(x)}{\sqrt{x}}\,dx$$

We try integration by parts with

$w = \ln(x)$ $\qquad dw = \frac{1}{x}\,dx$

$dv = x^{-1/2}\,dx$ $\qquad v = 2\sqrt{x}$

Now we can write

$$\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx = \lim_{T\to 0^+}\left[\left[2\sqrt{x}\ln(x)\right]_T^1 - \int_T^1 2\sqrt{x}\cdot\frac{1}{x}\,dx\right]$$

$$= \lim_{T\to 0^+}\left[\left[2\sqrt{x}\ln(x)\right]_T^1 - 2\int_T^1 x^{-1/2}\,dx\right]$$

$$= \lim_{T\to 0^+}\left[2\sqrt{x}\ln(x) - 2\cdot\frac{x^{1/2}}{1/2}\right]_T^1$$

$$= \lim_{T\to 0^+}\left[2\sqrt{x}\ln(x) - 4\sqrt{x}\right]_T^1$$

$$= \lim_{T\to 0^+}\left[(2\cdot 1\cdot 0 - 4\cdot 1) - (2\sqrt{T}\ln(T) - 4\sqrt{T})\right]$$

$$= \lim_{T\to 0^+}\left[-4 - 2\sqrt{T}\ln(T) + 4\sqrt{T}\right]$$

$$= -4 - 2\lim_{T\to 0^+}\sqrt{T}\ln(T)$$

$$= -4 - 2\lim_{T\to 0^+}\frac{\ln(T)}{1/\sqrt{T}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$\overset{H}{=} -4 - 2\lim_{T\to 0^+}\frac{\frac{d}{dT}\left[\ln(T)\right]}{\frac{d}{dT}\left[T^{-1/2}\right]}$$

$$= -4 - 2\lim_{T\to 0^+}\frac{1/T}{-\frac{1}{2}T^{-3/2}}$$

$$= -4 - 2\lim_{T\to 0^+}(-2\sqrt{T})$$

$$= -4 + 4\lim_{T\to 0^+}\sqrt{T}$$

$$= -4 + 0 \boxed{= -4} \quad (\text{convergent})$$