Section 3.3: Trigonometric Substitution

<u>Target</u>: Integrals involving the radical forms
$\sqrt{x^2+k^2}$, $\sqrt{k^2-x^2}$ or $\sqrt{x^2-k^2}$ where
k is assumed to be a positive constant.

Our approach is to eliminate the square root by replacing x with an appropriate expression in terms of a new variable θ.

eg $\sqrt{x^2+1}$

We let $x = \tan(\theta)$ so we have

$$x^2+1 = \tan^2(\theta)+1$$
$$= \sec^2(\theta)$$

Then $\sqrt{x^2+1} = \sqrt{\sec^2(\theta)}$
$$= \sec(\theta)$$

This is called a <u>trigonometric substitution</u>:

When the integral involves the form $\sqrt{x^2+k^2}$, we use the substitution $x = k\tan(\theta)$. Then

$$\sqrt{x^2+k^2} = \sqrt{k^2\tan^2(\theta)+k^2} = \sqrt{k^2\left[\tan^2(\theta)+1\right]}$$

$$= \sqrt{k^2\sec^2(\theta)} = k\sec(\theta)$$

This also means that $dx = \frac{d}{d\theta}\left[k\tan(\theta)\right]d\theta$

$$= k\sec^2(\theta)\,d\theta.$$

eg $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

Here $k=1$ so we set $x=\tan(\theta)$ so $dx = \sec^2(\theta)d\theta$.

Furthermore, $\sqrt{x^2+1} = \sec(\theta)$.

The integral becomes

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sec(\theta)}\cdot \sec^2(\theta)\,d\theta$$

$$= \int \sec(\theta)\,d\theta$$

$$= \ln\left|\sec(\theta)+\tan(\theta)\right| + C$$

$$\boxed{= \ln\left|\sqrt{x^2+1}+x\right| + C}$$

When the integral involves the form $\sqrt{k^2-x^2}$, we use the substitution $x = k\sin(\theta)$. Then

$$\sqrt{k^2-x^2} = \sqrt{k^2-k^2\sin^2(\theta)} = \sqrt{k^2[1-\sin^2(\theta)]}$$
$$= \sqrt{k^2\cos^2(\theta)} = k\cos(\theta).$$

Also, $dx = k\cos(\theta)\,d\theta$.

eg $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

Here $k=2$ so we set $x = 2\sin(\theta)$ so $dx = 2\cos(\theta)\,d\theta$.

We have
$$\sqrt{4-x^2} = \sqrt{4-4\sin^2(\theta)}$$
$$= \sqrt{4[1-\sin^2(\theta)]}$$
$$= \sqrt{4\cos^2(\theta)} = 2\cos(\theta)$$
$$x^2 = 4\sin^2(\theta)$$

The integral becomes

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{[4\sin^2(\theta)]\cdot[2\cos(\theta)]}\cdot 2\cos(\theta)\,d\theta$$
$$= \frac{1}{4}\int \frac{1}{\sin^2(\theta)}\,d\theta$$
$$= \frac{1}{4}\int \csc^2(\theta)\,d\theta$$
$$= \frac{1}{4}\left[-\cot(\theta)\right] + C$$

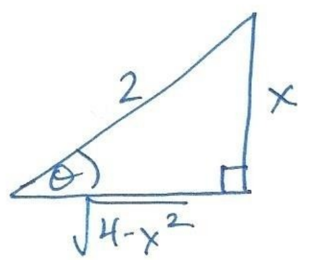
$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\cot(\theta) + C$$

We know that $x = 2\sin(\theta)$ so $\sin(\theta) = \frac{1}{2}x$

$\sqrt{4-x^2} = 2\cos(\theta)$ so $\cos(\theta) = \frac{1}{2}\sqrt{4-x^2}$

and thus $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{\frac{1}{2}\sqrt{4-x^2}}{\frac{1}{2}x} = \dfrac{\sqrt{4-x^2}}{x}$

Alternatively, we can use $\sin(\theta) = \frac{1}{2}x = \frac{x}{2}$ to create a right triangle with included angle θ, opposite side of length x, and hypotenuse of length 2.



By the Pythagorean theorem, the adjacent side is of length $\sqrt{4-x^2}$.

Again, this tells us that $\cot(\theta) = \dfrac{\sqrt{4-x^2}}{x}$.

Either way, $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x} + C$

$$\boxed{= -\frac{\sqrt{4-x^2}}{4x} + C}$$

When the integral involves the form $\sqrt{x^2-k^2}$, we use the substitution $x = k\sec(\theta)$. Then

$$\sqrt{x^2-k^2} = \sqrt{k^2\sec^2(\theta) - k^2} = \sqrt{k^2\left[\sec^2(\theta) - 1\right]}$$

$$= \sqrt{k^2\tan^2(\theta)} = k\tan(\theta).$$

Here, $dx = k\sec(\theta)\tan(\theta)\,d\theta$.

eg $\displaystyle\int \frac{\sqrt{x^2-5}}{x}\,dx$

Since $k=\sqrt{5}$, we set $x = \sqrt{5}\sec(\theta)$

$dx = \sqrt{5}\sec(\theta)\tan(\theta)\,d\theta$

and $\sqrt{x^2-5} = \sqrt{5\sec^2(\theta)-5} = \sqrt{5\left[\sec^2(\theta)-1\right]}$

$= \sqrt{5\tan^2(\theta)} = \sqrt{5}\tan(\theta)$.

The integral becomes

$$\int \frac{\sqrt{x^2-5}}{x}\,dx = \int \frac{\sqrt{5}\tan(\theta)}{\sqrt{5}\sec(\theta)}\cdot\sqrt{5}\sec(\theta)\tan(\theta)\,d\theta$$

$$= \sqrt{5}\int \tan^2(\theta)\,d\theta$$

$$= \sqrt{5}\int \left[\sec^2(\theta)-1\right]d\theta$$

$$= \sqrt{5}\left[\tan(\theta)-\theta\right]+C$$

$$= \sqrt{5}\tan(\theta) - \sqrt{5}\,\theta + C$$

$$= \sqrt{5}\cdot\frac{\sqrt{x^2-5}}{\sqrt{5}} - \sqrt{5}\cdot\operatorname{arcsec}\left(\frac{\sqrt{5}x}{5}\right)+C$$

$$\boxed{= \sqrt{x^2-5} - \sqrt{5}\operatorname{arcsec}\left(\frac{\sqrt{5}x}{5}\right)+C}$$

Since $\sec(\theta) = \frac{x}{\sqrt{5}} = \frac{\sqrt{5}x}{5}$

$\theta = \operatorname{arcsec}\left(\frac{\sqrt{5}x}{5}\right)$

We can sometimes apply trigonometric substitution even when an integral does not immediately appear to include a square root.

$$\text{eg} \quad \int \frac{x^3}{(x^2+4)^3}\,dx = \int \frac{x^3}{\left(\sqrt{x^2+4}\right)^6}\,dx$$

We let $x = 2\tan(\theta)$ so $dx = 2\sec^2(\theta)\,d\theta$.

Then $\sqrt{x^2+4} = \sqrt{4\tan^2(\theta)+4} = 2\sec(\theta)$

$$\left(\sqrt{x^2+4}\right)^6 = 64\sec^6(\theta)$$

$$x^3 = 8\tan^3(\theta)$$

The integral becomes

$$\int \frac{x^3}{(x^2+4)^3}\,dx = \int \frac{8\tan^3(\theta)}{64\sec^6(\theta)} \cdot 2\sec^2(\theta)\,d\theta$$

$$= \frac{1}{4}\int \frac{\tan^3(\theta)}{\sec^4(\theta)}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sin^3(\theta)}{\cos^3(\theta)} \cdot \cos^4(\theta)\,d\theta$$

Let $u = \sin(\theta)$

$du = \cos(\theta)\,d\theta$

$$= \frac{1}{4}\int \sin^3(\theta)\cos(\theta)\,d\theta$$

$$= \frac{1}{4}\int u^3\,du$$

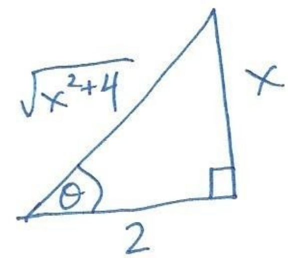
$$= \frac{1}{4}\left[\frac{u^4}{4}\right] + C$$

$$= \frac{1}{16}u^4 + C$$

Now we have

$$\int \frac{x^3}{(x^2+4)^3}\,dx = \frac{1}{16}\sin^4(\theta) + C$$

Since $x = 2\tan(\theta)$, $\tan(\theta) = \frac{x}{2}$ so we construct an appropriate right triangle:



Thus $\sin(\theta) = \dfrac{x}{\sqrt{x^2+4}}$ and so

$$\int \frac{x^3}{(x^2+4)^3}\,dx = \frac{1}{16}\cdot\left(\frac{x}{\sqrt{x^2+4}}\right)^4 + C$$

$$\boxed{= \frac{x^4}{16(x^2+4)^2} + C}$$

Integrals requiring trigonometric substitution often require strategies for trigonometric integrals as well.

eg $\int \sqrt{4-x^2}\, dx$

We let $x = 2\sin(\theta)$ so $du = 2\cos(\theta)\, d\theta$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2(\theta)} = 2\cos(\theta)$$

Now we have

$$\int \sqrt{4-x^2}\, dx = \int 2\cos(\theta) \cdot 2\cos(\theta)\, d\theta$$

$$= 4\int \cos^2(\theta)\, d\theta$$

$$= 4\int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= 2\int \left[1+\cos(2\theta)\right] d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}\cdot 2\sin(\theta)\cos(\theta) + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$$

$$\boxed{= 2\arcsin\left(\frac{x}{2}\right) + \frac{1}{4}x\sqrt{4-x^2} + C}$$

As with u-substitution, in order to apply the method of trigonometric substitution to a definite integral, the bounds must also be transformed into the corresponding values of the new variable θ. However, there is also no need to transform back into the original variable after integrating: we simply apply FTC ② in terms of θ.

eg $\displaystyle\int_0^{\sqrt{3}} \frac{1}{(x^2+9)^{3/2}}\,dx$

Note that $\displaystyle\int_0^{\sqrt{3}} \frac{1}{(x^2+9)^{3/2}}\,dx = \int_0^{\sqrt{3}} \frac{1}{(\sqrt{x^2+9})^3}\,dx$

We let $x = 3\tan(\theta)$ so $dx = 3\sec^2(\theta)\,d\theta$

$$\sqrt{x^2+9} = \sqrt{9\tan^2(\theta)+9} = 3\sec(\theta)$$

$$(\sqrt{x^2+9})^3 = [3\sec(\theta)]^3 = 27\sec^3(\theta)$$

When $x=0$, we have $3\tan(\theta) = 0$

$$\tan(\theta) = 0$$

$$\theta = \arctan(0) = 0$$

When $x=\sqrt{3}$, we have $3\tan(\theta) = \sqrt{3}$

$$\tan(\theta) = \frac{\sqrt{3}}{3}$$

$$\theta = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

The integral becomes

$$\int_0^{\sqrt{3}} \frac{1}{(x^2+9)^{3/2}}\,dx = \int_0^{\pi/6} \frac{1}{27\sec^3(\theta)} \cdot 3\sec^2(\theta)\,d\theta$$

Now we have

$$\int_0^{\sqrt{3}} \frac{1}{(x^2+9)^{3/2}}\,dx = \frac{1}{9}\int_0^{\pi/6} \frac{1}{\sec(\theta)}\,d\theta$$

$$= \frac{1}{9}\int_0^{\pi/6} \cos(\theta)\,d\theta$$

$$= \frac{1}{9}\Big[\sin(\theta)\Big]_0^{\pi/6}$$

$$= \frac{1}{9}\left[\sin\left(\frac{\pi}{6}\right) - \sin(0)\right] = \frac{1}{9}\cdot\frac{1}{2} \boxed{= \frac{1}{18}}$$