Section 3.1: The Method of Partial Fractions

Target: Integrals of proper rational functions

Reminder: We can always rewrite an improper rational function as the sum of a polynomial and a proper rational function, using long division

Some proper rational functions give ~~are~~ elementary integrals such as $\int \frac{1}{x^2}\,dx$, $\int \frac{1}{x}\,dx$ or $\int \frac{1}{x^2+1}\,dx$.

Others yield combinations of elementary integrals, such as $\int \frac{x+1}{x^2}\,dx$. Still others can be integrated using u-substitution, such as $\int \frac{x}{x^2+1}\,dx$.

However, others cannot be integrated using any of these methods.

$$\text{eg} \int \frac{1}{x^2-4}\,dx$$

Fortunately, a proper rational function can normally be decomposed into a sum of partial fractions, which are proper rational functions where the denominator is of a smaller degree than the original denominator.

$$\text{eg} \quad \frac{1}{x^2-4} = \frac{1/4}{x-2} - \frac{1/4}{x+2}$$

The denominator of each partial fraction must be a factor of the original denominator, so we will always start finding the partial fraction decomposition by factoring the denominator of the given proper rational function.

Thus the only denominators we need to consider for a partial fraction are the linear and irreducible quadratic factors of the original denominator, which may be unique or repeated n times.

This gives 4 cases:

1. Suppose $ax+b$ is a unique linear factor of the original denominator. Then each such factor yields a term in the partial fraction decomposition of the form
$$\frac{A}{ax+b} \quad \text{for some constant } A.$$

eg 1. $$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

2. $$\frac{1}{x^3+3x^2+2x} = \frac{1}{x(x^2+3x+2)}$$
$$= \frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

② Suppose $ax+b$ is a linear factor of the original denominator that is repeated n times, so that it has the form $(ax+b)^n$. Then each such factor yields n terms in the partial fraction decomposition of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$

where $A_1, A_2, A_3, \ldots, A_n$ are constants.

eg $$\frac{x+5}{x^2+2x+1} = \frac{x+5}{(x+1)^2}$$

$$= \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

③ Suppose ax^2+bx+c is a unique irreducible quadratic factor of the original denominator. Then each such factor yields a term in the partial fraction decomposition of the form

$$\frac{Ax+B}{ax^2+bx+c} \quad \text{for constants } A, B$$

eg $$\frac{7x^3}{x^4-1} = \frac{7x^3}{(x^2-1)(x^2+1)} = \frac{7x^3}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Note that a factor of x^2 can be treated as a unique irreducible quadratic factor, yielding a term in the partial fraction decomposition of the form

$$\frac{Ax+B}{x^2}.$$

But it can also be viewed as the linear factor x repeated twice, giving terms of the form

$$\frac{A}{x} + \frac{B}{x^2}$$

Observe that $\frac{x}{x} \cdot \frac{A}{x} + \frac{B}{x^2} = \frac{Ax}{x^2} + \frac{B}{x^2} = \frac{Ax+B}{x^2}$

(4) Suppose ax^2+bx+c is an irreducible quadratic factor of the original denominator that is repeated n times, so that it has the form $(ax^2+bx+c)^n$. Then each such factor yields n terms in the partial fraction decomposition of the form

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3}$$

$$+ \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$ are constants.

eg $$\frac{5x+6}{(x^2+4)^3} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3}$$

To begin solving for the constants in the partial fraction decomposition, we first multiply both sides of the expression by the original denominator.

Then, one approach is to substitute appropriate values of x into the resulting equation and solve for the constants.

eg $\dfrac{1}{x^2-4} = \dfrac{1}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$

Now we multiply both sides of the equation by the original denominator:

$$\frac{1}{(x-2)(x+2)} \cdot (x-2)(x+2) = \left(\frac{A}{x-2} + \frac{B}{x+2}\right) \cdot (x-2)(x+2)$$

$$1 = A(x+2) + B(x-2)$$

For $x=2$, we have

$$1 = A(2+2) + B(2-2)$$

$$1 = 4A \rightarrow A = 1/4$$

For $x=-2$, we have

$$1 = A(-2+2) + B(-2-2)$$

$$1 = -4B \rightarrow B = -1/4$$

Then we see that

$$\frac{1}{x^2-4} = \frac{1/4}{x-2} + \frac{-1/4}{x+2} \boxed{= \frac{1/4}{x-2} - \frac{1/4}{x+2}}$$

Now $\displaystyle\int \frac{1}{x^2-4}\,dx = \int \left[\frac{1/4}{x-2} - \frac{1/4}{x+2}\right] dx$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

We can apply the method of partial fractions to a definite integral in the usual way.

eg $\displaystyle\int_1^4 \frac{9x+5}{3x^2+x}\,dx$

We decompose into partial fractions:

$$\frac{9x+5}{3x^2+x} = \frac{9x+5}{x(3x+1)} = \frac{A}{x} + \frac{B}{3x+1}$$

$$9x+5 = A(3x+1) + Bx$$

For $x=0$,

$$5 = A\cdot 1 + B\cdot 0 \rightarrow A=5$$

For $x=-1/3$,

$$2 = A\cdot 0 + B\cdot\left(-\tfrac{1}{3}\right)$$

$$2 = -\tfrac{1}{3}B \quad\rightarrow\quad B=-6$$

Hence $\displaystyle\frac{9x+5}{3x^2+x} = \frac{5}{x} - \frac{6}{3x+1}$

$$\int_1^4 \frac{9x+5}{3x^2+x}\,dx = \int_1^4 \left(\frac{5}{x} - \frac{6}{3x+1}\right)dx$$

$$= \Big[5\ln|x| - 2\ln|3x+1|\Big]_1^4$$

$$\boxed{= 7\ln(4) - 2\ln(13)}$$

We cannot apply the method of partial fractions directly to an improper rational function, but may do so after performing long division.

eg $\displaystyle\int \frac{x^4+4x-3}{x^3-x^2}\,dx$

$$\begin{array}{r l}
 & x+1 \\
x^3-x^2 \,\big) & x^4 \qquad\quad +4x-3 \\
 & \underline{x^4-x^3} \\
 & \quad x^3 \quad +4x-3 \\
 & \quad \underline{x^3-x^2} \\
 & \qquad x^2+4x-3
\end{array}$$

Now we can write $\dfrac{x^4+4x-3}{x^3-x^2} = x+1+\dfrac{x^2+4x-3}{x^3-x^2}$

and apply the method of partial fractions:

$$\frac{x^2+4x-3}{x^3-x^2} = \frac{x^2+4x-3}{x^2(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$$

$$x^2+4x-3 = Ax(x-1)+B(x-1)+Cx^2$$

For $x=0$, $-3=-B \rightarrow B=3$

For $x=1$, $2=C \rightarrow C=2$

For $x=-1$, $-6 = 2A-2B+C$

$-6 = 2A-6+2$

$-2=2A \rightarrow A=-1$

Now we can write

$$\int \frac{x^4+4x-3}{x^3-x^2}\,dx = \int \left((x+1) - \frac{1}{x} + \frac{3}{x^2} + \frac{2}{x-1} \right) dx$$

$$= \frac{x^2}{2} + x - \ln|x| + 3\cdot\frac{x^{-1}}{-1} + 2\ln|x-1| + C$$

$$\boxed{= \frac{1}{2}x^2 + x - \ln|x| - \frac{3}{x} + 2\ln|x-1| + C}$$

The process of finding the constants in the partial fraction decomposition becomes more cumbersome with more repeated and / or irreducible quadratic factors of the denominator.

eg $\displaystyle\int \frac{7x^2-7x+1}{x^3-2x^2+x-2}\,dx$

Note that $x^3-2x^2+x-2 = x^2(x-2)+(x-2)$

$$= (x-2)(x^2+1)$$

so $\displaystyle\frac{7x^2-7x+1}{x^3-2x^2+x-2} = \frac{7x^2-7x+1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$

$$7x^2-7x+1 = A(x^2+1) + (Bx+C)(x-2)$$

For $x=2$, $\quad 15 = 5A \rightarrow A = 3$

$x=0$, $\quad 1 = 3\cdot 1 + C\cdot(-2)$

$\quad -2 = -2C \rightarrow C = 1$

$x=1$, $\quad 1 = 3\cdot 2 + (B+1)\cdot(-1)$

$\quad -5 = -B-1 \rightarrow B = 4$

Now we have

$$\int \frac{7x^2-7x+1}{x^3-2x^2+x-2}\,dx = \int \left(\frac{3}{x-2} + \frac{4x+1}{x^2+1} \right) dx$$

$$= \int \left(\frac{3}{x-2} + \frac{4x}{x^2+1} + \frac{1}{x^2+1} \right) dx$$

Let $u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$= 3\ln|x-2| + \arctan(x) + 4\int \frac{x}{x^2+1}\,dx$$

$$= 3\ln|x-2| + \arctan(x) + 2\int \frac{1}{u}\,du$$

$$= 3\ln|x-2| + \arctan(x) + 2\ln|u| + C$$

$$\boxed{= 3\ln|x-2| + \arctan(x) + 2\ln(x^2+1) + C}$$