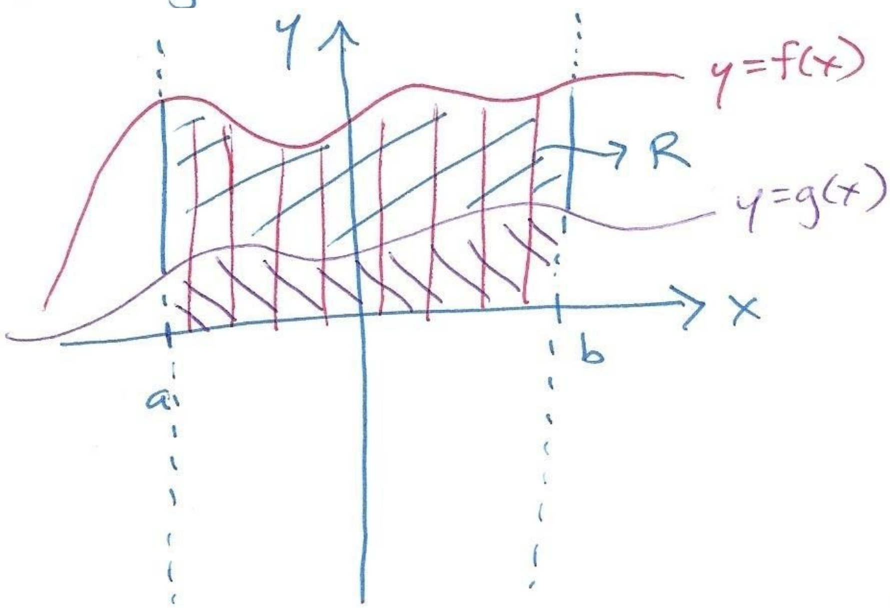
Section 2.4: Area Between Curves

Now we will try to find the area A of a region R which is bounded above by the curve $y=f(x)$, below by the curve $y=g(x)$, to the left by $x=a$, and to the right $x=b$.



Let R_1 be the region under $y=f(x)$ on $[a,b]$, and let A_1 be its area.

Let R_2 be the region under $y=g(x)$ on $[a,b]$, and let A_2 be its area.

Then
$$
\begin{aligned}
A &= A_1 - A_2 \\
&= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\
&= \int_a^b [f(x)-g(x)]\,dx
\end{aligned}
$$

This is the formula for the area between curves. Note that we must have both $f(x)$ and $g(x)$ continuous on $[a,b]$ and $f(x) \geq g(x)$ on $[a,b]$.

Note that, if $g(x) \equiv 0$ then the bottom boundary curve is the line $y=0$, the x-axis. Then the area between curves formula becomes

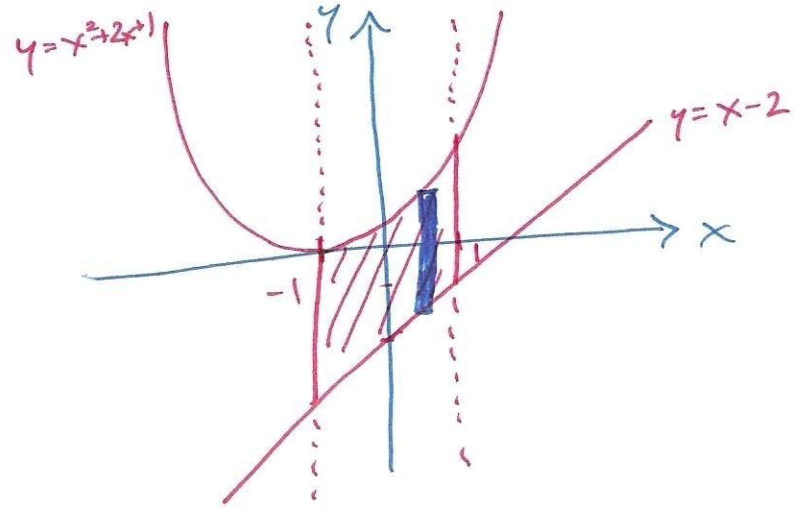
$$A = \int_a^b [f(x) - 0]\, dx = \int_a^b f(x)\, dx$$

where $f(x) \geq 0$ on $[a,b]$. This is the area under a curve formula again.

Given two boundary curves, how do we identify $f(x)$ and $g(x)$?

1. Sketch the graph.
2. Check for any points of intersection on $[a,b]$ by setting the functions equal to each other. If there are none, choose any point $x=p$ on $[a,b]$ and evaluate the functions there. The function that gives the larger value is $f(x)$. The other is $g(x)$.

eg Find the area of the region bounded by $y = x^2 + 2x + 1$ and $y = x - 2$ on the interval $[-1, 1]$.



Observe that

$$y = x^2 + 2x + 1 = (x+1)^2.$$

We can see that, on $[-1,1]$,

$$x^2 + 2x + 1 \geq x - 2$$

so $f(x) = x^2 + 2x + 1$

$g(x) = x - 2$.

Alternatively we could identify $f(x)$ and $g(x)$ by first setting

$$x^2 + 2x + 1 = x - 2$$

$$x^2 + x + 3 = 0 \rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{-11}}{2}$$

This equation has no solution so the two curves have no points of intersection.

Thus we check a point like $x = 0$:

$$x^2 + 2x + 1 = 1 \qquad x - 2 = -2$$

so $x^2 + 2x + 1 \geq x - 2$ on $[-1, 1]$ so $f(x) = x^2 + 2x + 1$

$g(x) = x - 2$.

$$A = \int_{-1}^{1} \left[(x^2+2x+1) - (x-2)\right] dx$$

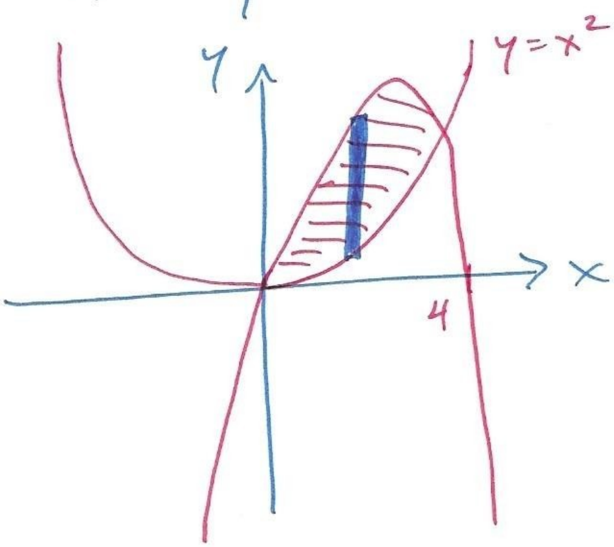
$$= \int_{-1}^{1} (x^2 + x + 3)\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 3x\right]_{-1}^{1}$$

$$= \left(\frac{1}{3} + \frac{1}{2} + 3\right) - \left(\frac{-1}{3} + \frac{1}{2} - 3\right) \boxed{= \frac{20}{3}}$$

Sometimes, the top and bottom boundary curves will form a natural region via their points of intersection, which means that we don't always have to specify an interval $[a, b]$.

eg Find the area of the region between $y = x^2$ and $y = 4x - x^2$.



We must solve for the points of intersection, because they will be the bounds on the definite integral for A.

We set

$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2x(x-2) = 0$$

$$x = 0 \quad x = 2$$

Thus we can see from the graph that, on $[0, 2]$,

$$f(x) = 4x - x^2$$

$$g(x) = x^2$$

$$A = \int_0^2 \left[(4x - x^2) - x^2 \right] dx$$

$$= \int_0^2 (4x - 2x^2)\, dx$$

$$= \left[4 \cdot \frac{x^2}{2} - 2 \cdot \frac{x^3}{3} \right]_0^2$$

$$= \left[2x^2 - \frac{2}{3}x^3 \right]_0^2$$

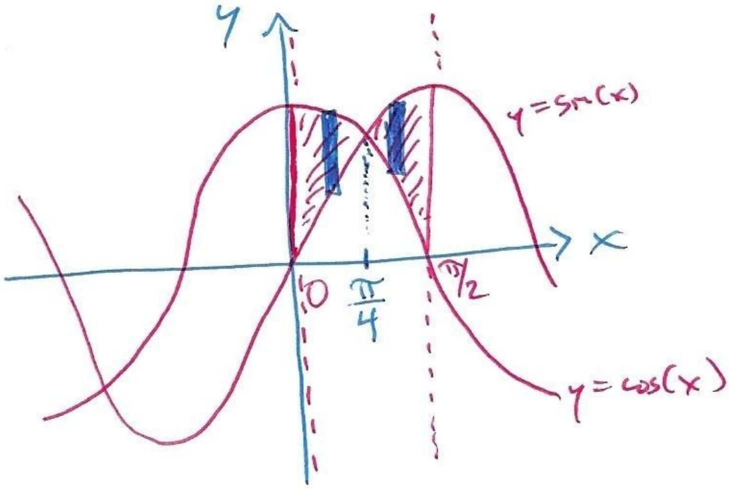
$$= \left[8 - \frac{16}{3} \right] - \left[0 - 0 \right] \boxed{= \frac{8}{3}}$$

The regions considered so far have been <u>vertically simple</u>. If we sketch a representative, vertically-oriented rectangle anywhere in the region, its top would always be defined by the same curve $y = f(x)$, and its bottom by the same curve $y = g(x)$.

What if a region <u>isn't</u> vertically simple?

One possibility is that we may be able to divide it into two or more regions that <u>are</u> vertically simple. Then we can use the area between curves formula to find the area of each vertically simple sub-region and add them together to get the area of the entire region.

eg Find the area of the region between $y = \sin(x)$ and $y = \cos(x)$ on $[0, \pi/2]$.



We set

$$\sin(x) = \cos(x)$$

$$\tan(x) = 1$$

$$x = \arctan(1) = \frac{\pi}{4}$$

So now we consider the vertically simple region on $[0, \pi/4]$ where $f(x) = \cos(x)$ and $g(x) = \sin(x)$.

Then the area A_1 of this region is

$$A_1 = \int_0^{\pi/4} [\cos(x) - \sin(x)]\, dx$$

$$= \Big[\sin(x) + \cos(x)\Big]_0^{\pi/4}$$

$$= \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] - [0 + 1] = \sqrt{2} - 1$$

Next we consider the vertically simple region on $[\pi/4, \pi/2]$ where $f(x) = \sin(x)$ and $g(x) = \cos(x)$.

Then the area A_2 of this region is

$$A_2 = \int_{\pi/4}^{\pi/2} [\sin(x) - \cos(x)]\, dx$$

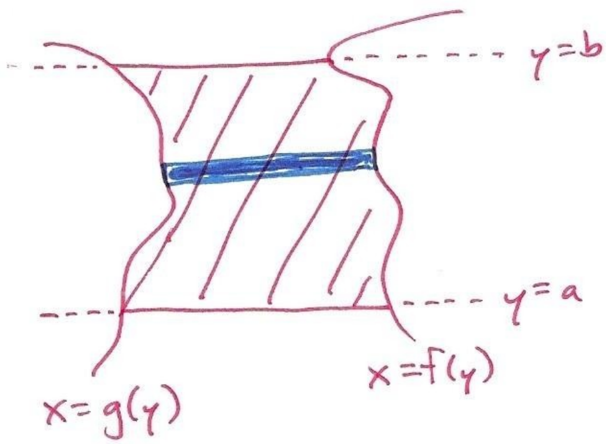
$$A_2 = \Big[-\cos(x) - \sin(x)\Big]_{\pi/4}^{\pi/2}$$

$$= \Big[0 - 1\Big] - \Big[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\Big]$$

$$= -1 + \sqrt{2} \quad = \sqrt{2} - 1$$

The area A of the entire region on $[0, \pi/2]$ is

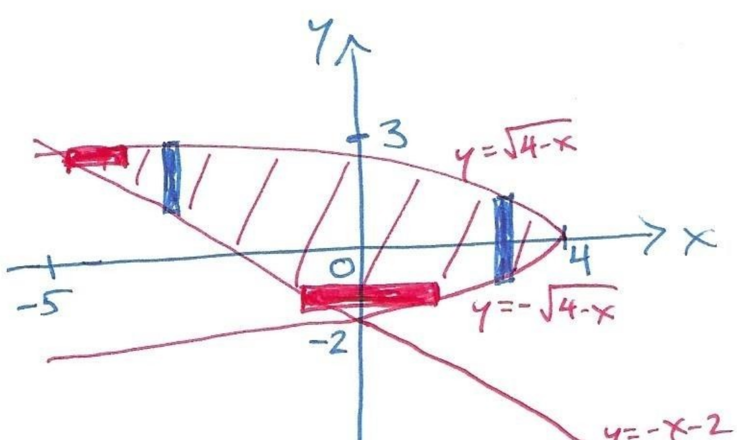
$$A = A_1 + A_2 = (\sqrt{2}-1) + (\sqrt{2}-1) \boxed{= 2\sqrt{2} - 2}.$$



Now consider a region bounded by a curve $x=f(y)$ to the right, $x=g(y)$ to the left, and the horizontal lines $y=a$ and $y=b$ on bottom and on top. Then this a <u>horizontally simple</u> region, and can be approximated using horizontally-oriented rectangles. Therefore the area A of this region is

$$A = \int_a^b \left[f(y) - g(y) \right] dy$$

eg Find the area of the region bounded by $y=-x-2$, $y=\sqrt{4-x}$ and $y=-\sqrt{4-x}$.



This region is not vertically simple, but can be divided into two vertically simple regions.

We need to find where

$$-x-2=\sqrt{4-x}$$
$$(-x-2)^2 = 4-x$$
$$x^2+4x+4 = 4-x$$
$$x^2+5x = 0$$
$$x(x+5)=0 \rightarrow \underset{\text{OMIT}}{x=0}, \; x=-5$$

The first vertically simple region lies on $[-5, 0]$ with $f(x) = \sqrt{4-x}$ and $g(x) = -x-2$. Its area is

$$A_1 = \int_{-5}^{0} \left[\sqrt{4-x} - (-x-2)\right] dx$$

$$= \int_{-5}^{0} \left(\sqrt{4-x} + x + 2\right) dx$$

$$= \left[\frac{(4-x)^{3/2}}{-3/2} + \frac{1}{2}x^2 + 2x\right]_{-5}^{0}$$

$$= \left[-\frac{2}{3}(4-x)^{3/2} + \frac{1}{2}x^2 + 2x\right]_{-5}^{0} = \frac{61}{6}$$

The second vertically simple region lies on $[0, 4]$ with $f(x) = \sqrt{4-x}$ and $g(x) = -\sqrt{4-x}$. Its area is

$$A_2 = \int_{0}^{4} \left[\sqrt{4-x} - (-\sqrt{4-x})\right] dx$$

$$= 2\int_{0}^{4} \sqrt{4-x}\, dx$$

$$= 2\left[\frac{(4-x)^{3/2}}{-3/2}\right]_{0}^{4} = \frac{32}{3}$$

The area A of the entire region is $A = \frac{61}{6} + \frac{32}{3} \boxed{= \frac{125}{6}}$

We can rewrite $y = \pm\sqrt{4-x}$ as

$$y^2 = 4-x$$

$$x = 4-y^2$$

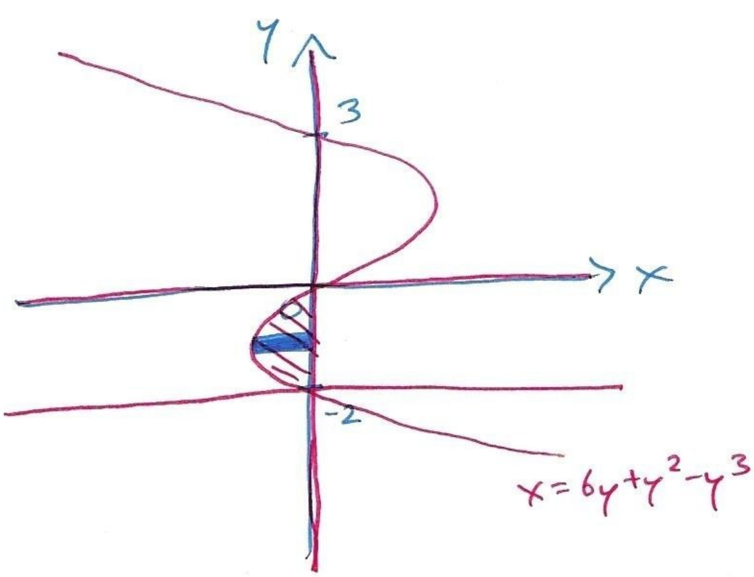
and $y=-x-2$ as $x=-y-2$. As a horizontally simple region, then, the region is bounded by $f(y)=4-y^2$ and $g(y)=-y-2$ on the y-interval $[-2,3]$. Hence

$$A = \int_{-2}^{3} \left[(4-y^2) - (-y-2) \right] dy$$

$$= \int_{-2}^{3} (6+y-y^2)\, dy$$

$$= \left[6y + \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 \right]_{-2}^{3} \boxed{= \frac{125}{6}}$$

eg Find the area of the region bounded by $x = 6y+y^2-y^3$, $y=-2$, and the x- and y-axes.



We can write

$$x = 6y+y^2-y^3$$

$$= -y(y^2-y-6)$$

$$= -y(y-3)(y+2)$$

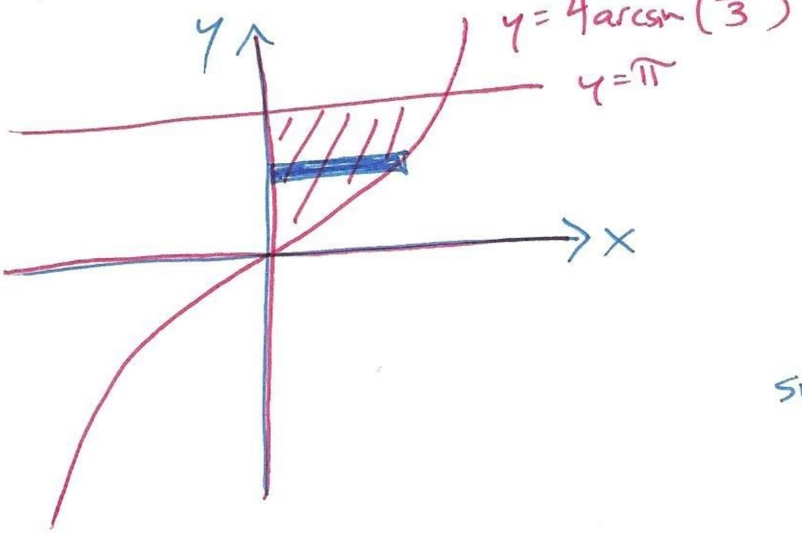
This is a horizontally simple region with $f(y)=0$, $g(y) = 6y+y^2-y^3$ on $[-2,0]$.

$$A = \int_{-2}^{0} \left[0 - (6y + y^2 - y^3)\right] dy$$

$$= \int_{-2}^{0} (y^3 - y^2 - 6y)\, dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{3} - 3y^2\right]_{-2}^{0} \boxed{= \frac{16}{3}}$$

eg Find the area of the region between $y = 4\arcsin\left(\frac{x}{3}\right)$, $y = \pi$, $y = 0$ and $x = 0$.



We can rewrite

$$y = 4\arcsin\left(\frac{x}{3}\right)$$

$$\frac{y}{4} = \arcsin\left(\frac{x}{3}\right)$$

$$\sin\left(\frac{y}{4}\right) = \frac{x}{3}$$

$$x = 3\sin\left(\frac{y}{4}\right)$$

While this region is both vertically and horizontally simple, it appears that the integration will be simpler in terms of y:

$$A = \int_{0}^{\pi} \left[3\sin\left(\frac{y}{4}\right) - 0\right] dy$$

$$= 3\int_{0}^{\pi} \sin\left(\frac{y}{4}\right) dy$$

$$= 3\left[\frac{-\cos(y/4)}{1/4}\right]_{0}^{\pi}$$

$$= -12\left[\cos\left(\frac{y}{4}\right)\right]_{0}^{\pi} \boxed{= 12 - 6\sqrt{2}}$$