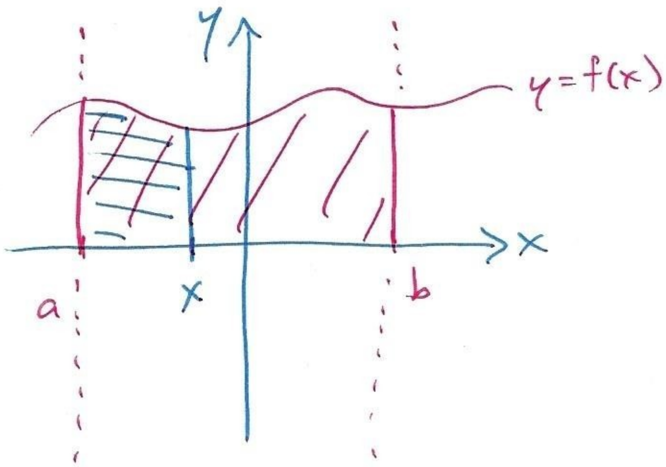
Section 2.3: The Fundamental Theorem of Calculus

Recall that if $f(x)$ is continuous and non-negative on $[a,b]$ then the area A of the region under the curve is given by

$$A = \int_a^b f(x)\,dx.$$

We could rewrite this in terms of any other variable of integration, such as

$$A = \int_a^b f(t)\,dt.$$



Now consider a point x on $[a,b]$. Then the area A_1 of the region under $y=f(x)$ on $[a,x]$ is given by

$$A_1 = \int_a^x f(t)\,dt.$$

Then this defines a function of x, which can be written

$$g(x) = \int_a^x f(t)\,dt.$$

For instance, in optics we study Fresnel functions like

$$S(x) = \int_0^x \sin(t^2)\,dt.$$

Now suppose we have a small constant h and consider both $g(x)$ and $g(x+h)$. Then $g(x+h)-g(x)$ represents the area under $y=f(x)$ which lies on the interval $[a, x+h]$ but not on $[a,x]$. This can be approximated as a rectangle of width h and height $f(x)$. Thus

$$g(x+h)-g(x) \approx hf(x)$$

$$\frac{g(x+h)-g(x)}{h} \approx f(x)$$

This approximation will become more and more accurate as $h \to 0$, so

$$f(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$$

Theorem: The First Fundamental Theorem (FTC ①)

If $f(t)$ is continuous on $[a,b]$ then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and differentiable on (a,b).

Furthermore, $g'(x) = f(x)$.

eg Given $S(x) = \int_0^x \sin(t^2)\,dt$

$$S'(x) = \sin(x^2).$$

If the upper bound is a function of x, rather than just x itself, we apply FTC ① with the Chain Rule.

eg $g(x) = \int_1^{3x^2} e^{t^4}\,dt$

$$g'(x) = e^{(3x^2)^4} \cdot [3x^2]'$$

$$= e^{81x^8} \cdot 6x$$

$$\boxed{= 6x\,e^{81x^8}}$$

eg $y = \int_{-2}^{\ln(x)} \cos(t^3)\,dt$

$$\frac{dy}{dx} = \cos(\ln^3(x)) \cdot \frac{d}{dx}[\ln(x)]$$

$$\boxed{= \frac{\cos(\ln^3(x))}{x}}$$

Given a function of the form

$$g(x) = \int_x^a f(t)\,dt$$

then we apply FTC ① by rewriting it as

$$g(x) = -\int_a^x f(t)\,dt.$$

eg $$g(x) = \int_x^3 \arcsin\left(\frac{1}{t^2}\right)dt$$

$$= -\int_3^x \arcsin\left(\frac{1}{t^2}\right)dt$$

$$g'(x) = -\left[\int_3^x \arcsin\left(\frac{1}{t^2}\right)dt\right]'$$

$$\boxed{= -\arcsin\left(\frac{1}{x^2}\right)}$$

Given a function of the form

$$g(x) = \int_{a(x)}^{b(x)} f(t)\,dt$$

we apply the Additive Interval Property and rewrite it as

$$g(x) = \int_{a(x)}^{0} f(t)\,dt + \int_0^{b(x)} f(t)\,dt$$

$$= -\int_0^{a(x)} f(t)\,dt + \int_0^{b(x)} f(t)\,dt$$

eg $g(x) = \int_x^{x^4} \tan(\sqrt{t})\,dt$

$$= \int_x^0 \tan(\sqrt{t})\,dt + \int_0^{x^4} \tan(\sqrt{t})\,dt$$

$$= -\int_0^x \tan(\sqrt{t})\,dt + \int_0^{x^4} \tan(\sqrt{t})\,dt$$

$$g'(x) = -\tan(\sqrt{x}) + \tan(\sqrt{x^4}) \cdot [x^4]'$$

$$\boxed{= -\tan(\sqrt{x}) + 4x^3 \tan(x^2)}$$

Theorem: The Second Fundamental Theorem (FTC ②)

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

eg $\int_{-2}^{1} x^3\,dx$

An antiderivative of x^3 is $\frac{x^4}{4}$ so, by FTC ②,

$$\int_{-2}^{1} x^3\,dx = \frac{1^4}{4} - \frac{(-2)^4}{4} = \frac{1}{4} - 4 \boxed{= -\frac{15}{4}}$$

Notationally, we can write

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Proof: Recall that the Mean Value Theorem (MVT) states that if $z(x)$ is continuous and differentiable on $[c,d]$ then there exists a point $x=p$ on (c,d) for which

$$z'(p) = \frac{z(d)-z(c)}{d-c}.$$

Now observe that $F(x)$ satisfies the MVT on $[a,b]$ and, therefore, on any of the n subintervals into which $[a,b]$ can be partitioned.

So consider one such subinterval $[x_{i-1}, x_i]$. By the MVT, there exists a point $x=p_i$ on (x_{i-1}, x_i) for which

$$F'(p_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

$$= \frac{F(x_i) - F(x_{i-1})}{\Delta x_i}$$

Furthermore, since $F(x)$ is an antiderivative of $f(x)$, then $F'(x) = f(x)$. So now we can write

$$f(p_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x_i}$$

$$f(p_i)\Delta x_i = F(x_i) - F(x_{i-1})$$

Now recall that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

If we set the sample point $x_i^* = p_i$ then we have

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(p_i)\Delta x_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1})\right]\end{aligned}$$

Observe that

$$\begin{aligned}&\sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1})\right] \\ &= \left[F(x_1) - F(x_0)\right] + \left[F(x_2) - F(x_1)\right] \\ &\quad + \left[F(x_3) - F(x_2)\right] + \left[F(x_4) - F(x_3)\right] \\ &\quad + \cdots + \left[F(x_{n-1}) - F(x_{n-2})\right] + \left[F(x_n) - F(x_{n-1})\right] \\ &= F(x_n) - F(x_0) \\ &= F(b) - F(a)\end{aligned}$$

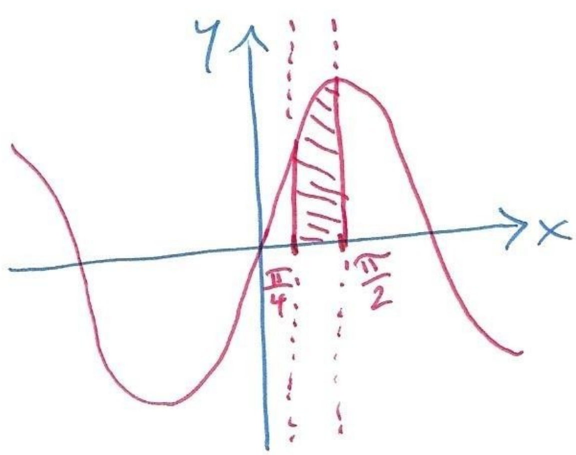
Now we can write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[F(b)-F(a)\right]$$
$$= F(b) - F(a).$$

eg
$$\int_0^{3/2} \frac{1}{\sqrt{3-x^2}}\,dx = \left[\arcsin\left(\frac{x}{\sqrt{3}}\right)\right]_0^{3/2}$$
$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(0)$$
$$= \frac{\pi}{3} - 0 \quad \boxed{= \frac{\pi}{3}}$$

We can apply FTC 2 to find areas as well.

eg Find the area under $y = \sin(x)$ from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$.



$$A = \int_{\pi/4}^{\pi/2} \sin(x)\,dx$$
$$= \left[-\cos(x)\right]_{\pi/4}^{\pi/2}$$
$$= -\cos\left(\frac{\pi}{2}\right) - \left[-\cos\left(\frac{\pi}{4}\right)\right]$$
$$= 0 + \frac{\sqrt{2}}{2}$$
$$\boxed{= \frac{\sqrt{2}}{2}}$$

We can apply all of the methods for computing indefinite integrals to the evaluation of definite integrals using FTC 2.

eg $\int_0^4 \sqrt{x}\,(x-2)\,dx$

$= \int_0^4 \left(x^{3/2} - 2x^{1/2}\right) dx$

$= \int_0^4 x^{3/2}\,dx - 2\int_0^4 x^{1/2}\,dx$

$= \left[\dfrac{x^{5/2}}{5/2}\right]_0^4 - 2\left[\dfrac{x^{3/2}}{3/2}\right]_0^4$

$= \frac{2}{5}\left[4^{5/2} - 0^{5/2}\right] - \frac{4}{3}\left[4^{3/2} - 0^{3/2}\right]$

$= \frac{2}{5}\left[32 - 0\right] - \frac{4}{3}\left[8 - 0\right]$

$= \frac{64}{5} - \frac{32}{3}$ $\boxed{= \frac{32}{15}}$

eg $\int_1^7 \dfrac{x+3}{3x}\,dx$

$= \int_1^7 \left(\frac{1}{3} + \frac{1}{x}\right) dx$

$= \left[\frac{1}{3}x + \ln|x|\right]_1^7$

$= \left[\frac{7}{3} + \ln(7)\right] - \left[\frac{1}{3} + \ln(1)\right]$

$\boxed{= 2 + \ln(7)}$

Given a definite integral $\int_a^b f(x)\,dx$ for which $f(x)$ is discontinuous on $[a,b]$, we call it an <u>improper</u> integral and FTC ② does <u>not</u> apply.

eg $\int_{-1}^{7} \frac{x+3}{3x}\,dx$ is improper because $\frac{x+3}{3x}$ is discontinuous at $x=0$, and $x=0$ lies on $[-1,7]$.

Sometimes we combine FTC ② with the Additive Interval Property.

eg $\int_1^6 |4-x|\,dx$ Recall $|4-x| = \begin{cases} 4-x, & x \leq 4 \\ -(4-x), & x>4 \end{cases}$

$$= \int_1^4 |4-x|\,dx + \int_4^6 |4-x|\,dx$$

$$= \int_1^4 (4-x)\,dx + \int_4^6 -(4-x)\,dx$$

$$= \left[4x - \frac{x^2}{2}\right]_1^4 - \left[4x - \frac{x^2}{2}\right]_4^6$$

$$= \left[(16-8)-(4-\tfrac{1}{2})\right] - \left[(24-18)-(16-8)\right]$$

$$\boxed{= \frac{13}{2}}$$

Why does $F(x)$ need only be an antiderivative of $f(x)$, and not the indefinite integral? Observe that

$$\left[F(x)+C\right]_a^b = \left[F(b)+C\right]-\left[F(a)+C\right]$$
$$= F(b)+C-F(a)-C$$
$$= F(b)-F(a)$$

To apply integration by parts to a definite integral we just rewrite the integration by parts formula as

$$\int_a^b w\,dv = \left[vw\right]_a^b - \int_a^b v\,dw$$

eg $\displaystyle\int_1^3 \frac{\ln(x)-1}{x^2}\,dx$

We use integration by parts with

$w=\ln(x)-1$ $\qquad dw=\frac{1}{x}\,dx$

$dv=\frac{1}{x^2}\,dx$ $\qquad v=-\frac{1}{x}$

So $$\int_1^3 \frac{\ln(x)-1}{x^2}\,dx = \left[-\frac{1}{x}(\ln(x)-1)\right]_1^3 + \int_1^3 \frac{1}{x^2}\,dx$$
$$= \left[-\frac{\ln(x)}{x}+\frac{1}{x}\right]_1^3 + \left[-\frac{1}{x}\right]_1^3$$

$$= \left[-\frac{\ln(x)}{x} \right]_1^3$$

$$= -\frac{\ln(3)}{3} + \frac{\ln(1)}{1} \boxed{= -\frac{\ln(3)}{3}}$$

For a definite integral $\int_a^b f(x)\,dx$, the bounds a and b are values of x. When applying u-substitution, then, we must find the corresponding u-values for $x=a$ and $x=b$, and use them as the new bounds of integration. However, we do not need to revert everything back to x after integrating: FTC ② can be applied directly to the definite integral written in terms of u.

eg $\int_0^2 3x^2(x^3+1)^3\,dx$

Let $u = x^3+1$

$du = 3x^2\,dx$

Thus, when $x=0$, $u = 0^3+1 = 1$

$x=2$, $u = 2^3+1 = 9$

Now the integral becomes

$$\int_0^2 3x^2(x^3+1)^3\,dx = \int_1^9 u^3\,du$$

$$= \left[\frac{u^4}{4}\right]_1^9 = \frac{9^4}{4} - \frac{1^4}{4} \boxed{= 1640}$$

eg $\int_0^{\pi/3} \sec^6(\theta)\tan(\theta)\,d\theta$

$= \int_0^{\pi/3} \sec^5(\theta)\cdot \sec(\theta)\tan(\theta)\,d\theta$

Let $u = \sec(\theta)$

$du = \sec(\theta)\tan(\theta)\,d\theta$

When $\theta = 0$, $u = \sec(0) = 1$

$\theta = \frac{\pi}{3}$, $u = \sec\left(\frac{\pi}{3}\right) = 2$

The integral becomes

$$\int_0^{\pi/3} \sec^6(\theta)\tan(\theta)\,d\theta = \int_1^2 u^5\,du$$

$$= \left[\frac{u^6}{6}\right]_1^2$$

$$= \frac{2^6}{6} - \frac{1^6}{6} \boxed{= \frac{21}{2}}$$