Section 1.2: Integration by Substitution

Consider a line ℓ. Its slope is given by

$$m = \frac{\Delta y}{\Delta x}$$

where Δx represents the change in the x-coordinate and Δy represents the corresponding change in the y-coordinate. Now suppose that ℓ is the tangent line to a curve $y = f(x)$ at a point x. Then $m = f'(x)$ so

$$\frac{\Delta y}{\Delta x} = f'(x) \quad \longrightarrow \quad \Delta y = f'(x)\,\Delta x.$$

$$\Delta y = \frac{dy}{dx}\,\Delta x$$

We typically use Δx and Δy to represent a "large" change in the variable. To indicate an infinitesimal change, we instead represent this as dx or dy, and we call these <u>differentials</u>. Hence

$$dy = \frac{dy}{dx}\,dx$$

This lets us rewrite an integral given with respect to one variable in terms of a different variable, because it shows us how to rewrite the differential.

Recall that the Chain Rule (for derivatives) is given by

$$[f(g(x))]' = f'(g(x))\, g'(x)$$

We therefore have

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

We can let $u = g(x)$ so then $\frac{du}{dx} = g'(x)$.

The integral can be rewritten as

$$\int f'(u) \cdot \frac{du}{dx} \cdot dx = f(u) + C$$

But $du = \frac{du}{dx} \cdot dx$ so this becomes

$$\int f'(u)\, du = f(u) + C$$

This is called <u>integration by substitution</u> or <u>u-substitution</u>.

eg $\int 2x\sqrt{x^2+1}\, dx$

We let $u = x^2 + 1$

$$\frac{du}{dx} = 2x \rightarrow du = 2x\, dx$$

Thus we can write

$$\int 2x\sqrt{x^2+1}\, dx = \int \sqrt{x^2+1} \cdot 2x\, dx = \int \sqrt{u}\, du$$

$$\int 2x\sqrt{x^2+1}\,dx = \int u^{1/2}\,du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$

$$\boxed{= \frac{2}{3}(x^2+1)^{3/2} + C}$$

General steps for u-substitution:

1. Identify an appropriate expression for u (often the inside function in a composite function)
2. Rewrite all expressions in the integral, including the differential, in terms of u.
3. Integrate the resulting expression, which will hopefully now be an elementary integral.
4. Substitute back in for u to write the answer in terms of the original variable.

eg $\int \cos(x)\, e^{\sin(x)}\,dx$

Let $u = \sin(x)$

$du = \cos(x)\,dx$

The integral becomes

$$\int \cos(x) e^{\sin(x)}\, dx = \int e^{u}\, du$$
$$= e^{u} + C$$
$$\boxed{= e^{\sin(x)} + C}$$

eg $\int 4\tan^3(\theta)\sec^2(\theta)\, d\theta$

Let $u = \tan(\theta)$

$du = \sec^2(\theta)\, d\theta$

The integral becomes

$$\int 4\tan^3(\theta)\sec^2(\theta)\, d\theta = \int 4u^3\, du$$
$$= 4\int u^3\, du$$
$$= 4\left[\frac{u^4}{4}\right] + C$$
$$\boxed{= \tan^4(\theta) + C}$$

eg $\int x^3 \cosh(x^4+2)\, dx$

Let $u = x^4 + 2$

$du = 4x^3\, dx$

But note that $\int x^3 \cosh(x^4+2)\, dx = \int \frac{1}{4} \cdot 4x^3 \cosh(x^4+2)\, dx$

More simply, we can write

$$\frac{1}{4}\,du = x^3\,dx$$

The integral becomes

$$\int x^3 \cosh(x^4+2)\,dx = \int \cosh(u)\cdot\frac{1}{4}\,du$$

$$= \frac{1}{4}\int \cosh(u)\,du$$

$$= \frac{1}{4}\left[\sinh(u)\right] + C$$

$$\boxed{= \frac{1}{4}\sinh(x^4+2) + C}$$

eg $\displaystyle\int \frac{5x^2}{\sqrt{1-2x^3}}\,dx$

We let $u = 1-2x^3$

$du = -6x^2\,dx$

$-\frac{1}{6}\,du = x^2\,dx$

The integral becomes

$$\int 5 \cdot \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{6}\,du\right) = -\frac{5}{6}\int u^{-1/2}\,du$$

$$= -\frac{5}{6}\left[\frac{u^{1/2}}{1/2}\right] + C$$

$$\boxed{= -\frac{5}{3}\sqrt{1-2x^3} + C}$$

eg $\displaystyle\int 3x^5\sqrt{1+x^3}\,dx$

Let $u = 1+x^3$

$du = 3x^2\,dx$

We can rewrite the integral as

$$\int 3x^5\sqrt{1+x^3}\,dx = \int x^3\sqrt{1+x^3}\cdot 3x^2\,dx$$

But observe that $x^3 = u-1$ so this becomes

$$\int (u-1)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right)du$$

$$\int 3x^5\sqrt{1+x^3}\,dx = \int u^{3/2}\,du - \int u^{1/2}\,du$$

$$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C$$

$$\boxed{= \frac{2}{5}(1+x^3)^{5/2} - \frac{2}{3}(1+x^3)^{3/2} + C}$$

Sometimes we can apply u-substitution to an integral which does not involve a composite function.

eg $\displaystyle\int \frac{\ln(x)}{x}\,dx = \int \ln(x)\cdot\frac{1}{x}\,dx$

Let $u = \ln(x)$

$du = \frac{1}{x}\,dx$

The integral becomes

$$\int \frac{\ln(x)}{x}\,dx = \int u\,du$$

$$= \frac{u^2}{2} + C \;\; \boxed{= \frac{[\ln(x)]^2}{2} + C = \frac{\ln^2(x)}{2} + C}$$

Recall that we have previously shown that if $\int f(x)dx = F(x) + C$

then $$\int f(mx+b)\,dx = \frac{1}{m}F(mx+b) + C$$

if $m \neq 0$. We can prove this using u-substitution.

We let $u = mx + b$

$$du = m\,dx$$

$$\frac{1}{m}\,du = dx$$

The integral becomes

$$\int f(u)\cdot\frac{1}{m}\,du = \frac{1}{m}\int f(u)\,du = \frac{1}{m}F(u) + C$$

$$= \frac{1}{m}F(mx+b) + C.$$

eg $\displaystyle\int \frac{2}{5x+9}\,dx$ Let $u = 5x+9$

$du = 5\,dx \rightarrow \frac{1}{5}\,du = dx$

$$= \int \frac{2}{u}\cdot\frac{1}{5}\,du$$

$$= \frac{2}{5}\int\frac{1}{u}\,du$$

$$= \frac{2}{5}\cdot\ln|u| + C \boxed{= \frac{2}{5}\ln|5x+9| + C}$$

In general when integrating a rational function, if we want to apply u-substitution then we will typically let u be the denominator.

eg $\displaystyle\int \frac{3x^2+x+4}{2x^3+x^2+8x+7}\,dx$

Let $u = 2x^3 + x^2 + 8x + 7$

$$du = (6x^2 + 2x + 8)\,dx$$

$$\frac{1}{2}\,du = (3x^2 + x + 4)\,dx$$

The integral becomes

$$\int \frac{3x^2+x+4}{2x^3+x^2+8x+7}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C$$

$$\boxed{= \frac{1}{2}\ln|2x^3+x^2+8x+7| + C}$$

In general, we can divide rational functions into <u>proper</u> and <u>improper</u> rational functions. They all have the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. In a proper rational function, the degree of $P(x)$ is <u>strictly less</u> than the degree of $Q(x)$. Otherwise, it is improper.

eg $f(x) = \dfrac{x^2+1}{x-3}$ is an improper rational function

$f(x) = \dfrac{3x^2+x+4}{2x^3+x^2+8x+7}$ is a proper rational function

$f(x) = \dfrac{5x+4}{2-3x}$ is an improper rational function

Improper rational functions can be rewritten in terms of proper rational functions using <u>long division</u>.

eg $\int \frac{x^2+1}{x-3}\,dx$

Let $u = x-3$

$du = 1 \cdot dx = dx$

Then $x = u+3$

$x^2+1 = (u+3)^2+1 = u^2+6u+10$

The integral becomes

$$\int \frac{x^2+1}{x-3}\,dx = \int \frac{u^2+6u+10}{u}\,du$$

$$= \int u\,du + 6\int du + 10\int \frac{1}{u}\,du$$

$$= \frac{u^2}{2} + 6 \cdot u + 10 \cdot \ln|u| + C$$

$$\boxed{= \frac{1}{2}(x-3)^2 + 6(x-3) + 10\ln|x-3| + C}$$

eg (cont.) $\displaystyle\int \frac{x^2+1}{x-3}\,dx$

We will use long division to rewrite the improper rational function as the <u>sum</u> of a <u>polynomial</u> and a <u>proper</u> rational function.

$$
\begin{array}{r l}
 & \quad x + 3 \\
\text{①}\quad x-3 & \overline{\big)\, x^2 \qquad\quad + 1} \\
 & \underline{\ x^2 - 3x\qquad} \\
 & \qquad 3x + 1 \quad \text{④} \\
 & \qquad \underline{3x - 9} \\
 & \qquad\quad\ \ 10
\end{array}
$$

③ ②

① Write the dividend and divisor in descending order of powers of x

② What multiplies x to get x^2?

③ Multiply this expression by the divisor

④ Subtract this expression from the dividend to obtain the remainder

⑤ Repeat ②-④ with the remainder as the new dividend until its degree is smaller than that of the divisor

The quotient obtained is the desired polynomial. The proper rational function is the remainder, divided by the original denominator.

Thus,

$$\frac{x^2+1}{x-3} = x+3 + \frac{10}{x-3}$$

$$\int \frac{x^2+1}{x-3}\,dx = \int \left(x+3+\frac{10}{x-3}\right)dx$$

$$= \frac{1}{2}x^2 + 3x + 10\cdot\frac{\ln|x-3|}{1} + C$$

$$\boxed{= \frac{1}{2}x^2 + 3x + 10\ln|x-3| + C}$$

eg $\displaystyle\int \frac{6x^2-2x-3}{1-2x^2}\,dx$

By long division, we have

$$\begin{array}{r|l} & \quad -3 \\ \hline -2x^2+1 & 6x^2-2x-3 \\ & \underline{6x^2 \qquad -3} \\ & \qquad -2x \end{array}$$

Thus $\displaystyle\int \frac{6x^2-2x-3}{1-2x^2}\,dx = \int\left(-3 - \frac{2x}{1-2x^2}\right)dx$

$$= -3x - 2\int \frac{x}{1-2x^2}\,dx$$

Now let $u = 1-2x^2$

$du = -4x\,dx \longrightarrow -\frac{1}{4}\,du = x\,dx$

We have

$$-3x - 2\int \frac{1}{u}\cdot\left(-\frac{1}{4}\,du\right)$$

$$= -3x + \frac{1}{2}\int \frac{1}{u}\,du$$

$$= -3x + \frac{1}{2}\ln|u| + C$$

$$\boxed{= -3x + \frac{1}{2}\ln|1-2x^2| + C}$$

Theorem: ① $\int \tan(x)\,dx = -\ln|\cos(x)| + C$

② $\int \cot(x)\,dx = \ln|\sin(x)| + C$

Proof: ① Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ so

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx.$$

Let $u = \cos(x)$

$du = -\sin(x)\,dx \rightarrow -du = \sin(x)\,dx$

The integral becomes

$$\int \tan(x)\,dx = \int \frac{1}{u}\cdot(-du) = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos(x)| + C.$$

Note that we can instead write

$$\int \tan(x)\,dx = \ln\left|[\cos(x)]^{-1}\right| + C$$

$$= \ln|\sec(x)| + C$$

eg $\int \tan\left(\frac{x}{5}\right)dx = \dfrac{-\ln\left|\cos\left(\frac{x}{5}\right)\right|}{1/5} + C$

$$\boxed{= -5\ln\left|\cos\left(\frac{x}{5}\right)\right| + C}$$

Theorem: ① $\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$

② $\int \csc(x)\,dx = -\ln|\csc(x)+\cot(x)| + C$

$= \ln|\csc(x) - \cot(x)| + C$

Proof: ② We rewrite

$$\csc(x) = \csc(x)\cdot\frac{\csc(x)+\cot(x)}{\csc(x)+\cot(x)}$$

$$= \frac{\csc^2(x)+\csc(x)\cot(x)}{\csc(x)+\cot(x)}$$

so $$\int \csc(x)\,dx = \int \frac{\csc^2(x)+\csc(x)\cot(x)}{\csc(x)+\cot(x)}\,dx$$

Let $u = \csc(x)+\cot(x)$

$du = [-\csc(x)\cot(x) - \csc^2(x)]\,dx$

$-du = [\csc^2(x)+\csc(x)\cot(x)]\,dx$

The integral becomes

$$\int \csc(x)\,dx = \int \frac{1}{u}\cdot(-du)$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\csc(x)+\cot(x)| + C.$$