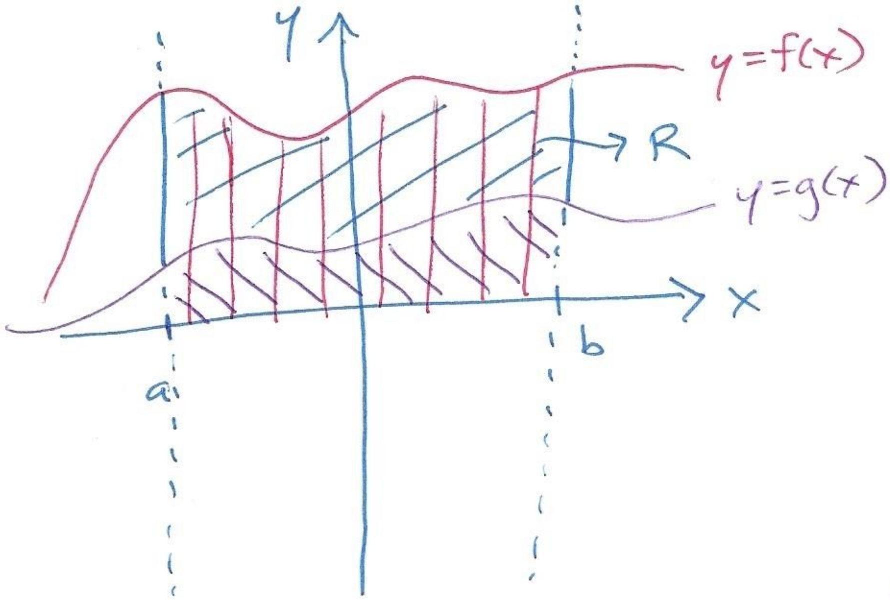


Let R_1 be the region under $y=f(x)$ on $[a,b]$, and let A_1 be its area.

Let R_2 be the region under $y=g(x)$ on $[a,b]$, and let A_2 be its area.

Then
$$\begin{aligned} A &= A_1 - A_2 \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b [f(x)-g(x)]\,dx \end{aligned}$$

This is the formula for the area between curves. Note that we must have both $f(x)$ and $g(x)$ continuous on $[a,b]$ and $f(x) \geq g(x)$ on $[a,b]$.

Note that, if $g(x) \equiv 0$ then the bottom boundary curve is the line $y=0$, the x-axis. Then the area between curves formula becomes
$$A = \int_a^b [f(x) - 0]\,dx = \int_a^b f(x)\,dx$$

where $f(x) \geq 0$ on $[a,b]$. This is the area under a curve formula again.

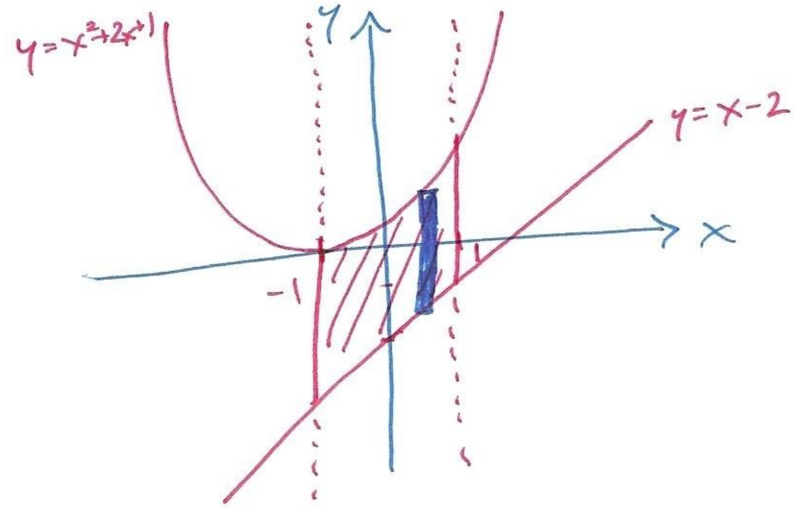
Given two boundary curves, how do we identify $f(x)$ and $g(x)$?

1. Sketch the graph.
2. Check for any points of intersection on $[a,b]$ by setting the functions equal to each other.
   If there are none, choose any point $x=p$ on $[a,b]$ and evaluate the functions there.
   The function that gives the larger value is $f(x)$.
   The other is $g(x)$.

Section 2.4

$$A = \int_a^b [f(x) - g(x)]\,dx$$

eg Find the area of the region bounded by $y = x^2 + 2x + 1$ and $y = x - 2$ on the interval $[-1, 1]$.



Observe that

$$y = x^2 + 2x + 1 = (x+1)^2.$$

We can see that, on $[-1, 1]$,

$$x^2 + 2x + 1 \geq x - 2$$

so $f(x) = x^2 + 2x + 1$

$g(x) = x - 2$.

Alternatively we could identify $f(x)$ and $g(x)$ by first setting

$$x^2 + 2x + 1 = x - 2$$

$$x^2 + x + 3 = 0 \quad \rightarrow \quad x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{-11}}{2}$$

This equation has no solution so the two curves have no points of intersection.

Thus we check a point like $x = 0$:

$$x^2 + 2x + 1 = 1 \qquad x - 2 = -2$$

so $x^2 + 2x + 1 \geq x - 2$ on $[-1, 1]$ so $f(x) = x^2 + 2x + 1$

$g(x) = x - 2$.

$$A = \int_{-1}^{1} \left[(x^2+2x+1) - (x-2)\right] dx$$

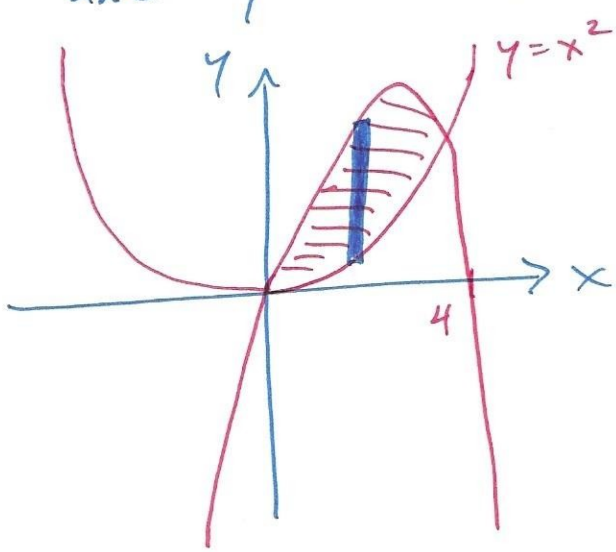
$$= \int_{-1}^{1} (x^2 + x + 3)\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 3x\right]_{-1}^{1}$$

$$= \left(\frac{1}{3} + \frac{1}{2} + 3\right) - \left(\frac{-1}{3} + \frac{1}{2} - 3\right) \boxed{= \frac{20}{3}}$$

Sometimes the top and bottom boundary curves will form a natural region via their points of intersection, which means that we don't always have to specify an interval $[a, b]$.

eg Find the area of the region between $y = x^2$ and $y = 4x - x^2$.



We must solve for the points of intersection, because they will be the bounds on the definite integral for A.

We set

$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2x(x-2) = 0$$

$$x = 0 \quad x = 2$$

Thus we can see from the graph that, on $[0, 2]$,

$$f(x) = 4x - x^2$$

$$g(x) = x^2$$

$$A = \int_0^2 \left[(4x - x^2) - x^2\right] dx$$

$$= \int_0^2 (4x - 2x^2)\, dx$$

$$= \left[4 \cdot \frac{x^2}{2} - 2 \cdot \frac{x^3}{3}\right]_0^2$$

$$= \left[2x^2 - \frac{2}{3}x^3\right]_0^2$$

$$= \left[8 - \frac{16}{3}\right] - \left[0 - 0\right] \boxed{= \frac{8}{3}}$$

The regions considered so far have been vertically simple. If we sketch a representative, vertically-oriented rectangle anywhere in the region, its top would always be defined by the same curve $y = f(x)$, and its bottom by the same curve $y = g(x)$.

What if a region isn't vertically simple?

One possibility is that we may be able to divide it into two or more regions that are vertically simple. Then we can use the area between curves formula to find the area of each vertically simple sub-region and add them together to get the area of the entire region.