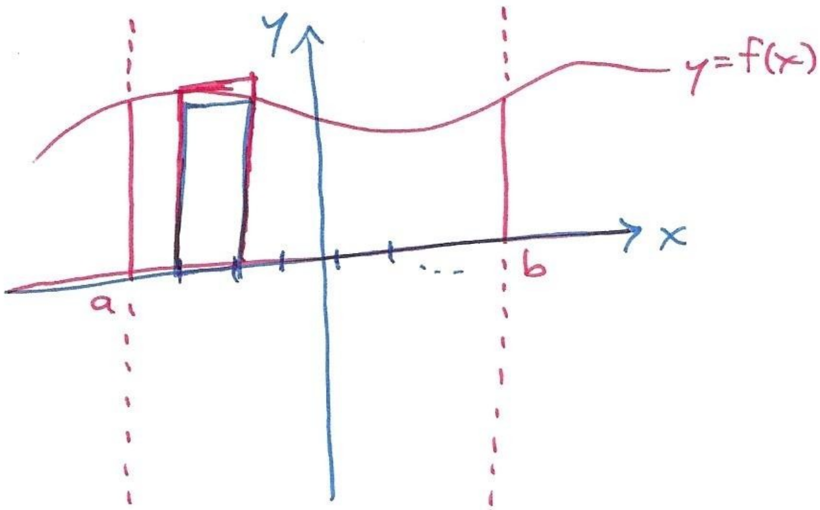
Section 2.1

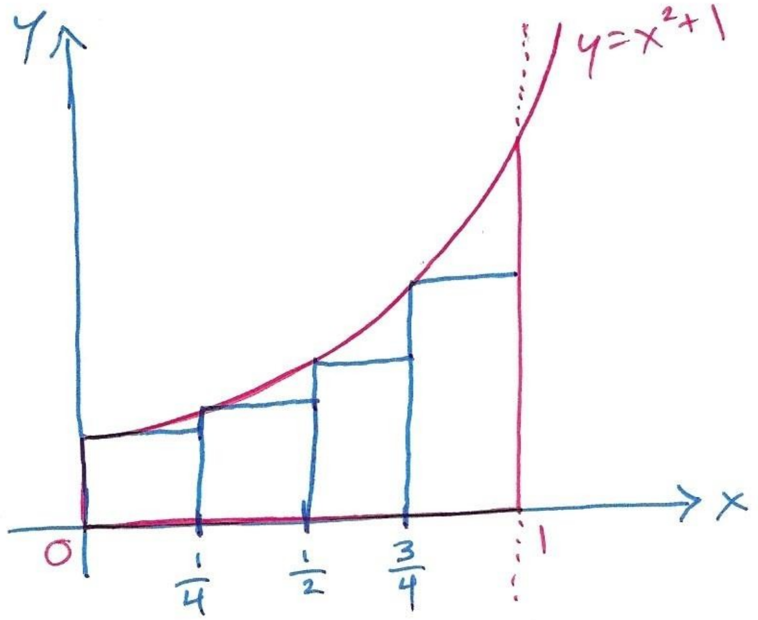


In order to find an approximation of the area A, we will approximate the region R with a number n of rectangles. Then we could compute the area of each rectangle, and estimate A by taking their sum.

We will divide the interval $[a,b]$ or $a \le x \le b$ into n <u>subintervals</u>, and use those subintervals as one side of each rectangle. We call this a <u>partition</u> of $[a,b]$.

If each subinterval has the same width Δx then we call it a <u>regular partition</u>.

eg Estimate the area bounded above by $f(x)=x^2+1$, below by the x-axis, to the left by $x=0$, and to the right by $x=1$. Do so with a regular partition into 4 subintervals.



We want to divide $[0,1]$ into 4 equal subintervals. Thus they will have width

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

which will also be the width of each rectangle.

Thus the first rectangle will be drawn on the subinterval $[0, \frac{1}{4}]$.

" second rectangle " $[\frac{1}{4}, \frac{1}{2}]$.

" third rectangle " $[\frac{1}{2}, \frac{3}{4}]$.

" fourth rectangle " $[\frac{3}{4}, 1]$.

For the height of each rectangle, we will choose the minimum value of $f(x)$ on the corresponding subinterval. These are called <u>inscribed</u> rectangles.

For the first rectangle, its height will be $f(0) = 1$, so its area is $A_1 = \frac{1}{4} \cdot 1 = \frac{1}{4}$.

For the second rectangle, its height will be $f(\frac{1}{4}) = \frac{17}{16}$, so its area is $A_2 = \frac{1}{4} \cdot \frac{17}{16} = \frac{17}{64}$.

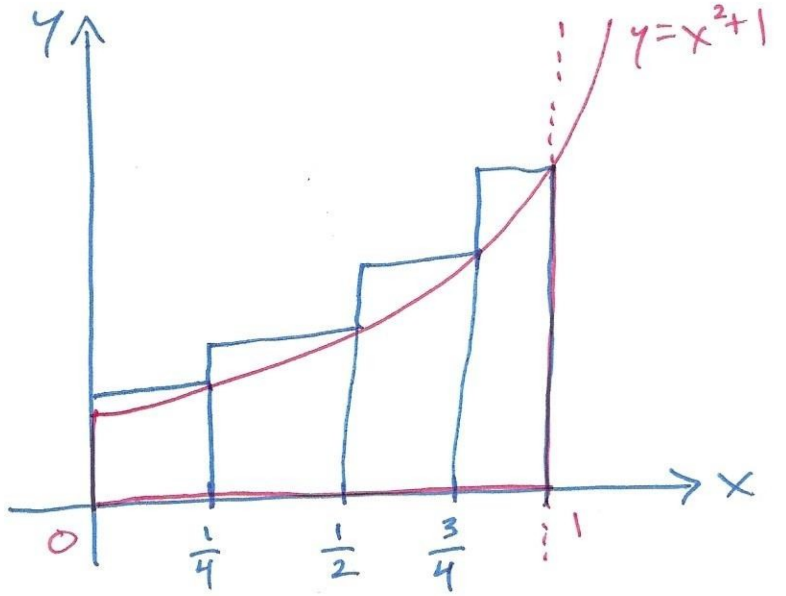
For the third rectangle, its height will be $f(\frac{1}{2}) = \frac{5}{4}$, so its area is $A_3 = \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}$.

For the fourth rectangle, its height will $f(\frac{3}{4}) = \frac{25}{16}$, so its area is $A_4 = \frac{1}{4} \cdot \frac{25}{16} = \frac{25}{64}$.

Then we can estimate the true area A to be

$$A \approx A_1 + A_2 + A_3 + A_4 = \frac{39}{32} \approx 1.22.$$

This is the smallest reasonable estimate of A, and is therefore called the <u>lower sum</u>.



Instead, we could try using the maximum value of $f(x)$ on the corresponding interval to give the height of each rectangle. These are called <u>circumscribed</u> rectangles.

For the first rectangle, its height will $f(\frac{1}{4}) = \frac{17}{16}$, so its area is $B_1 = \frac{1}{4} \cdot \frac{17}{16} = \frac{17}{64}$.

Likewise, the other rectangles will have heights of $f(\frac{1}{2}) = \frac{5}{4}$, $f(\frac{3}{4}) = \frac{25}{16}$ and $f(1) = 2$ so their areas will be

$$B_2 = \frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}, \quad B_3 = \frac{1}{4} \cdot \frac{25}{16} = \frac{25}{64}, \quad B_4 = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

Our new estimate of A is

$$A \approx B_1 + B_2 + B_3 + B_4 = \frac{47}{32} \approx 1.47.$$

This is the largest reasonable estimate of A, and is called the <u>upper sum</u>.

We can conclude that $\frac{39}{32} \leq A \leq \frac{47}{32}$.

Suppose we want to approximate A using n rectangles.
Then we partition the interval $[a, b]$ into n subintervals
of width

$$\Delta x = \frac{b-a}{n}.$$

Then the first subinterval is $[a, a+\Delta x]$

second subinterval is $[a+\Delta x, a+2\Delta x]$

$\vdots$

ith subinterval is $[a+(i-1)\Delta x, a+i\Delta x]$.

$\vdots$

nth subinterval is $[a+(n-1)\Delta x, a+n\Delta x]$

where $a+n\Delta x = a+n\cdot\dfrac{b-a}{n} = a+(b-a) = b.$

We call the right endpoint of the ith subinterval x_i

where

$$x_i = a+i\Delta x$$

Note that the left endpoint of the ith subinterval is,

therefore, x_{i-1}.

We can then denote $x_0 = a$ and $x_n = b$.

Now we assume that $f(x)$ is continuous on $[a, b]$ and therefore on each subinterval. Then the Extreme Value Theorem guarantees that $f(x)$ will have both a minimum value and a maximum value on each subinterval.

Let $x = m_i$ be the absolute minimum of $f(x)$ on the ith subinterval, so $f(m_i)$ is the minimum value. Then we would choose the height of each rectangle to be $f(m_i)$ in order to compute the lower sum.

Similarly let $x = M_i$ be the absolute maximum of $f(x)$ on the ith subinterval, so $f(M_i)$ is the maximum value. Then we choose $f(M_i)$ to be the height of the rectangles to obtain the upper sum.

In general, then, the lower sum is given by

$$s(n) = f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + \cdots + f(m_n)\Delta x$$
$$= [f(m_1) + f(m_2) + f(m_3) + \cdots + f(m_n)]\Delta x$$

The upper sum is given by

$$S(n) = f(M_1)\Delta x + f(M_2)\Delta x + f(M_3)\Delta x + \cdots + f(M_n)\Delta x$$
$$= [f(M_1) + f(M_2) + f(M_3) + \cdots + f(M_n)]\Delta x$$