MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 8 | **Mathematics 1001** | Winter 2024

SOLUTIONS

[5] 1. (a) Let $x = 5\sec(\theta)$ so $dx = 5\sec(\theta)\tan(\theta)\,d\theta$. Then $x^3 = 125\sec^3(\theta)$ and

$$\sqrt{x^2-25} = \sqrt{25\sec^2(\theta)-25} = 5\tan(\theta).$$

The integral becomes

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x^3}\,dx &= \int \frac{5\tan(\theta)}{125\sec^3(\theta)} \cdot 5\sec(\theta)\tan(\theta)\,d\theta \\ &= \frac{1}{5}\int \frac{\tan^2(\theta)}{\sec^2(\theta)}\,d\theta \\ &= \frac{1}{5}\int \sin^2(\theta)\,d\theta \\ &= \frac{1}{5}\int \left[\frac{1-\cos(2\theta)}{2}\right]d\theta \\ &= \frac{1}{10}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C \\ &= \frac{1}{10}\theta - \frac{1}{10}\sin(\theta)\cos(\theta) + C.\end{aligned}$$

Since $\sec(\theta) = \frac{x}{5}$, $\theta = \operatorname{arcsec}\left(\frac{x}{5}\right)$ and $\cos(\theta) = \frac{5}{x}$. Furthermore, we can draw a right triangle with interior angle θ, adjacent sidelength 5 and hypotenuse of length x. By the Pythagorean theorem, the opposite sidelength is $\sqrt{x^2-25}$ and so

$$\sin(\theta) = \frac{\sqrt{x^2-25}}{x}.$$

Finally, then,

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x^3}\,dx &= \frac{1}{10}\operatorname{arcsec}\left(\frac{x}{5}\right) - \frac{1}{10}\cdot\frac{\sqrt{x^2-25}}{x}\cdot\frac{5}{x} + C \\ &= \frac{1}{10}\operatorname{arcsec}\left(\frac{x}{5}\right) - \frac{\sqrt{x^2-25}}{2x^2} + C.\end{aligned}$$

[5] (b) Let $x = 4\sin(\theta)$ so $dx = 4\cos(\theta)\,d\theta$. Then

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2(\theta)} = \sqrt{16[1-\sin^2(\theta)]} = \sqrt{16\cos^2(\theta)} = 4\cos(\theta).$$

When $x = 0$, we have $0 = 4\sin(\theta)$ so $\theta = \arcsin(0) = 0$. When $x = 2\sqrt{2}$, we have $2\sqrt{2} = 4\sin(\theta)$ so $\theta = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$. Hence the integral becomes

$$\begin{aligned}\int_0^{2\sqrt{2}} \frac{x}{\sqrt{16-x^2}}\,dx &= \int_0^{\frac{\pi}{4}} \frac{4\sin(\theta)}{4\cos(\theta)} \cdot 4\cos(\theta)\,d\theta \\ &= 4\int_0^{\frac{\pi}{4}} \sin(\theta)\,d\theta \\ &= 4\Big[-\cos(\theta)\Big]_0^{\frac{\pi}{4}} \\ &= 4\left[-\cos\left(\frac{\pi}{4}\right) + \cos(0)\right] \\ &= 4\left(-\frac{\sqrt{2}}{2} + 1\right) \\ &= 4 - 2\sqrt{2}.\end{aligned}$$

[5] 2. (a) The integrand is discontinuous when $x^2 - 4 = 0$, so $x = \pm 2$. Since $x = 2$ is the upper bound of integration, we write

$$\int_0^2 \frac{1}{x^2-4}\,dx = \lim_{T\to 2^-} \int_0^T \frac{1}{x^2-4}\,dx.$$

In order to carry out the integration, we decompose the integrand into partial fractions:

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-2).$$

When $x = 2$, we have $1 = 4A$ so $A = \frac{1}{4}$. When $x = -2$, we obtain $1 = -4B$ so $B = -\frac{1}{4}$. Thus

$$\begin{aligned}\int_0^2 \frac{1}{x^2-4}\,dx &= \lim_{T\to 2^-} \int_0^T \left(\frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}\right) dx \\ &= \lim_{T\to 2^-} \left[\frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2|\right]_0^T \\ &= \frac{1}{4}\lim_{T\to 2^-} \left[\ln|T-2| - \ln|T+2| - \ln(2) + \ln(2)\right] \\ &= -\frac{1}{4}\ln(4) + \frac{1}{4}\lim_{T\to 2^-} \ln|T-2| \\ &= -\infty.\end{aligned}$$

Hence the improper integral is divergent.

[5] (b) We use integration by parts, with $w = x^2$ so $dw = 2x\,dx$, and $dv = e^{4x}\,dx$ so $v = \frac{1}{4}e^{4x}$. Then

$$\int_{-\infty}^{0} x^2 e^{4x}\,dx = \lim_{T\to-\infty} \int_{T}^{0} x^2 e^{4x}\,dx$$
$$= \lim_{T\to-\infty} \left(\left[\frac{1}{4}x^2 e^{4x} \right]_T^0 - \frac{1}{2}\int_T^0 x e^{4x}\,dx \right).$$

We use integration by parts again, with $w = x$ so $dw = dx$, and $dv = e^{4x}\,dx$ so $v = \frac{1}{4}e^{4x}$. Now

$$\int_{-\infty}^{0} x^2 e^{4x}\,dx = \lim_{T\to-\infty} \left(\left[\frac{1}{4}x^2 e^{4x} - \frac{1}{8}x e^{4x} \right]_T^0 + \frac{1}{8}\int_T^0 e^{4x}\,dx \right)$$
$$= \lim_{T\to-\infty} \left[\frac{1}{4}x^2 e^{4x} - \frac{1}{8}x e^{4x} + \frac{1}{32}e^{4x} \right]_T^0$$
$$= \lim_{T\to-\infty} \left[\frac{1}{32} - \frac{1}{4}T^2 e^{4T} + \frac{1}{8}T e^{4T} - \frac{1}{32}e^{4T} \right].$$

First note that

$$\lim_{T\to-\infty} e^{4T} = 0.$$

Next,

$$\lim_{T\to-\infty} T e^{4T} = \lim_{T\to-\infty} \frac{T}{e^{-4T}} \overset{\text{H}}{=} \lim_{T\to-\infty} \frac{1}{-4e^{-4T}} = -\frac{1}{4}\lim_{T\to-\infty} e^{4T} = 0.$$

Lastly,

$$\lim_{T\to-\infty} T^2 e^{4T} = \lim_{T\to-\infty} \frac{T^2}{e^{-4T}} \overset{\text{H}}{=} \lim_{T\to-\infty} \frac{2T}{-4e^{-4T}} = -\frac{1}{2}\lim_{T\to-\infty} \frac{T}{e^{-4T}}.$$

(Here we can use l'Hôpital's Rule a second time, or just apply our previous result.) So, finally,

$$\int_{-\infty}^{0} x^2 e^{4x}\,dx = \frac{1}{32} - 0 + 0 - 0 = \frac{1}{32}.$$