MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 8 | **MATHEMATICS 1001** | WINTER 2025

SOLUTIONS

[4] 1. (a) The radical is in the form $\sqrt{x^2+k^2}$ with $k=3$. We set $x = 3\tan(\theta)$ so $dx = 3\sec^2(\theta)\,d\theta$. Then

$$\sqrt{x^2+9} = \sqrt{9\tan^2(\theta)+9} = \sqrt{9\sec^2(\theta)} = 3\sec(\theta).$$

Now we have

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2+9}}\,dx &= \int \frac{1}{3\tan(\theta)\cdot 3\sec(\theta)}\cdot 3\sec^2(\theta)\,d\theta \\ &= \frac{1}{3}\int \frac{\sec(\theta)}{\tan(\theta)}\,d\theta \\ &= \frac{1}{3}\int \frac{1}{\cos(\theta)}\cdot\frac{\cos(\theta)}{\sin(\theta)}\,d\theta \\ &= \frac{1}{3}\int \csc(\theta)\,d\theta \\ &= -\frac{1}{3}\ln|\csc(\theta)+\cot(\theta)| + C.\end{aligned}$$

Since we already know that $\sqrt{x^2+9} = 3\sec(\theta)$, we can construct a right triangle with interior angle θ, adjacent side of length 3 and hypotenuse of length $\sqrt{x^2+9}$. Then the opposite sidelength is x, and so

$$\csc(\theta) = \frac{\sqrt{x^2+9}}{x} \quad \text{and} \quad \cot(\theta) = \frac{3}{x}.$$

This means that

$$\int \frac{1}{x\sqrt{x^2+9}}\,dx = -\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}}{x}+\frac{3}{x}\right| + C = -\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}+3}{x}\right| + C.$$

[6] (b) Since

$$\int_1^{\sqrt{2}} \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\,dx = \int_1^{\sqrt{2}} \frac{x^2}{\left(\sqrt{4-x^2}\right)^3}\,dx,$$

the radical is in the form $\sqrt{k^2-x^2}$ with $k=2$. We let $x = 2\sin(\theta)$ so $dx = 2\cos(\theta)\,d\theta$, $x^2 = 4\sin^2(\theta)$ and

$$\sqrt{4-x^2} = \sqrt{4-2\sin^2(\theta)} = \sqrt{4\cos^2(\theta)} = 2\cos(\theta).$$

When $x = 1$, $\sin(\theta) = \frac{1}{2}$ so $\theta = \frac{\pi}{6}$. When $x = \sqrt{2}$, $\sin(\theta) = \frac{\sqrt{2}}{2}$ so $\theta = \frac{\pi}{4}$. The integral becomes

$$\begin{aligned}
\int_1^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{4\sin^2(\theta)}{2\cos(\theta)} \cdot 2\cos(\theta)\,d\theta \\
&= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^2(\theta)\,d\theta \\
&= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1-\cos(2\theta)}{2}\,d\theta \\
&= 2\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\
&= 2\left[\frac{\pi}{4} - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right] \\
&= 2\left[\frac{\pi}{4} - \frac{1}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{4}\right] \\
&= \frac{\pi}{6} - 1 + \frac{\sqrt{3}}{2} \\
&= \frac{\pi - 6 + 3\sqrt{3}}{6}.
\end{aligned}$$

[5] 2. (a) The integrand has a discontinuity at $x = 0$, the lower bound of integration. Thus we write

$$\int_0^2 \ln\left(\frac{x}{2}\right)\,dx = \lim_{T\to 0^+} \int_T^2 \ln\left(\frac{x}{2}\right)\,dx.$$

Now we use integration by parts with $w = \ln\left(\frac{x}{2}\right)$ so $dw = \frac{1}{x}\,dx$ and $dv = dx$ so $v = x$. The integral becomes

$$\begin{aligned}
\int_0^2 \ln\left(\frac{x}{2}\right)\,dx &= \lim_{T\to 0^+}\left(\left[x\ln\left(\frac{x}{2}\right)\right]_T^2 - \int_T^2 dx\right) \\
&= \lim_{T\to 0^+}\left[x\ln\left(\frac{x}{2}\right) - x\right]_T^2 \\
&= \lim_{T\to 0^+}\left[2\ln(1) - 2 - T\ln\left(\frac{T}{2}\right) + T\right] \\
&= -2 - \lim_{T\to 0^+} T\ln\left(\frac{T}{2}\right).
\end{aligned}$$

The remaining limit is a $0 \cdot \infty$ indeterminate form, so we first rewrite the limit as an $\frac{\infty}{\infty}$

form and apply l'Hôpital's Rule:

$$\begin{aligned}\lim_{T\to 0^+} T\ln\left(\frac{T}{2}\right) &= \lim_{T\to 0^+} \frac{\ln\left(\frac{T}{2}\right)}{\frac{1}{T}} \\ &\stackrel{\text{H}}{=} \lim_{T\to 0^+} \frac{\frac{1}{T}}{-\frac{1}{T^2}} \\ &= \lim_{T\to 0^+}(-T) \\ &= 0.\end{aligned}$$

Thus

$$\int_0^2 \ln\left(\frac{x}{2}\right)\,dx = -2 - 0 = -2.$$

[5] (b) First we write

$$\int_3^\infty \frac{1}{x^2-4}\,dx = \lim_{T\to\infty}\int_3^T \frac{1}{x^2-4}\,dx.$$

In order to carry out the integration, we decompose the integrand into partial fractions:

$$\begin{aligned}\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} &= \frac{A}{x-2} + \frac{B}{x+2} \\ 1 &= A(x+2) + B(x-2).\end{aligned}$$

When $x = 2$, we have $1 = 4A$ so $A = \frac{1}{4}$. When $x = -2$, we obtain $1 = -4B$ so $B = -\frac{1}{4}$. Thus

$$\begin{aligned}\int_3^\infty \frac{1}{x^2-4}\,dx &= \lim_{T\to\infty}\int_3^T \left[\frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}\right] dx \\ &= \lim_{T\to\infty}\left[\frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2|\right]_3^T \\ &= \frac{1}{4}\lim_{T\to\infty}[\ln|T-2| - \ln|T+2| - \ln(1) + \ln(5)] \\ &= \frac{1}{4}\lim_{T\to\infty}\ln\left|\frac{T-2}{T+2}\right| + \frac{1}{4}\ln(5) \\ &= \frac{1}{4}\ln(1) + \frac{1}{4}\ln(5) \\ &= \frac{1}{4}\ln(5).\end{aligned}$$