MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 | **Mathematics 1001** | WINTER 2024

SOLUTIONS

[5] 1. (a) We can write

$$\frac{2x^2-4x-1}{3x^3+x^2+3x+1} = \frac{2x^2-4x-1}{(3x+1)(x^2+1)} = \frac{A}{3x+1} + \frac{Bx+D}{x^2+1}.$$

Thus

$$2x^2-4x-1 = A(x^2+1)+(Bx+D)(3x+1).$$

When $x=-\frac{1}{3}$, we have $\frac{5}{9} = A\cdot\frac{10}{9}$ so $A=\frac{1}{2}$. When $x=0$, we have $-1=A+D$ so $D=-1-A=-\frac{3}{2}$. Finally, when (say) $x=1$, we have $-3=2A+4B+4D=1+4B-6$ so $B=\frac{1}{2}$. Hence

$$\begin{aligned}\int \frac{2x^2-4x-1}{3x^3+x^2+3x+1} &= \int \left(\frac{\frac{1}{2}}{3x+1} + \frac{\frac{1}{2}x-\frac{3}{2}}{x^2+1}\right) dx \\ &= \frac{1}{2}\int \frac{1}{3x+1}\,dx + \frac{1}{2}\int\frac{x}{x^2+1}\,dx - \frac{3}{2}\int\frac{1}{x^2+1}\,dx \\ &= \frac{1}{6}\ln|3x+1| + \frac{1}{2}\int\frac{x}{x^2+1}\,dx - \frac{3}{2}\arctan(x).\end{aligned}$$

For the remaining integral, we let $u=x^2+1$ so $\frac{1}{2}\,du = x\,dx$. Then

$$\begin{aligned}\int \frac{2x^2-4x-1}{3x^3+x^2+3x+1} &= \frac{1}{6}\ln|3x+1| + \frac{1}{4}\int\frac{1}{u}\,du - \frac{3}{2}\arctan(x) \\ &= \frac{1}{6}\ln|3x+1| + \frac{1}{4}\ln|u| - \frac{3}{2}\arctan(x) + C \\ &= \frac{1}{6}\ln|3x+1| + \frac{1}{4}\ln(x^2+1) - \frac{3}{2}\arctan(x) + C.\end{aligned}$$

[6] (b) This is an improper rational function, so first we need to perform long division:

$$\begin{array}{r|l} & \phantom{2x^6 - 3x^4 + {}} 2x^2 - 3 \\ \hline x^4-1 & 2x^6 - 3x^4 + 9x^2 \\ & 2x^6 \phantom{{}- 3x^4} - 2x^2 \\ \hline & - 3x^4 + 11x^2 \\ & - 3x^4 \phantom{{}+ 11x^2} + 3 \\ \hline & 11x^2 - 3 \end{array}$$

Thus

$$\frac{2x^6 - 3x^4 + 9x^2}{x^4 - 1} = 2x^2 - 3 + \frac{11x^2 - 3}{x^4 - 1}.$$

Now we carry out the partial fraction decomposition on the remainder, such that

$$\frac{11x^2 - 3}{x^4 - 1} = \frac{11x^2 - 3}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Dx + E}{x^2+1}.$$

Therefore

$$11x^2 - 3 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Dx+E)(x-1)(x+1).$$

When $x = 1$ we have $8 = 4A$ so $A = 2$. When $x = -1$ we have $8 = -4B$ so $B = -2$. When $x = 0$ we have $-3 = A - B - E = 4 - E$ so $E = 7$. And when (say) $x = 2$ we have $41 = 15A + 5B + 6D + 3E = 30 - 10 + 6D + 21$ so $D = 0$. The integral becomes

$$\begin{aligned}\int \frac{2x^6 - 3x^4 + 9x^2}{x^4 - 1}\,dx &= \int \left[2x^2 - 3 + \frac{2}{x-1} - \frac{2}{x+1} + \frac{7}{x^2+1}\right] dx \\ &= \frac{2}{3}x^3 - 3x + 2\ln|x-1| - 2\ln|x+1| + 7\arctan(x) + C.\end{aligned}$$

[5] 2. (a) We write

$$\begin{aligned}\int \frac{\cos^5(3x)}{\sin^4(3x)}\,dx &= \int \frac{\cos^4(3x)}{\sin^4(3x)} \cdot \cos(3x)\,dx \\ &= \int \frac{[1 - \sin^2(3x)]^2}{\sin^4(3x)} \cdot \cos(3x)\,dx.\end{aligned}$$

Now we let $u = \sin(3x)$ so $du = 3\cos(3x)\,dx$ and $\frac{1}{3}\,du = \cos(3x)\,dx$. The integral becomes

$$\begin{aligned}\int \frac{\cos^5(3x)}{\sin^4(3x)}\,dx &= \frac{1}{3}\int \frac{[1-u^2]^2}{u^4}\,du \\ &= \frac{1}{3}\int \frac{1 - 2u^2 + u^4}{u^4}\,du \\ &= \frac{1}{3}\int (u^{-4} - 2u^{-2} + 1)\,du \\ &= \frac{1}{3}\left[\frac{u^{-3}}{-3} - 2 \cdot \frac{u^{-1}}{-1} + u\right] + C \\ &= -\frac{1}{9\sin^3(3x)} + \frac{2}{3\sin(3x)} + \frac{1}{3}\sin(3x) + C.\end{aligned}$$

[4] (b) Using both half-angle identities, we write

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\,dx &= \int \left[\frac{1-\cos(2x)}{2}\right]\cdot\left[\frac{1+\cos(2x)}{2}\right]dx\\ &= \frac{1}{4}\int [1-\cos^2(2x)]\,dx\\ &= \frac{1}{4}\int \sin^2(2x)\,dx.\end{aligned}$$

Now we apply the half-angle formula again to obtain

$$\begin{aligned}\int \sin^2(x)\cos^2(x)\,dx &= \frac{1}{4}\int \frac{1-\cos(4x)}{2}\,dx\\ &= \frac{1}{8}\left[x-\frac{1}{4}\sin(4x)\right]+C\\ &= \frac{1}{8}x-\frac{1}{32}\sin(4x)+C.\end{aligned}$$