MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 6 **MATHEMATICS 1001** WINTER 2025

SOLUTIONS

[2] 1. (a) See Figure 1.

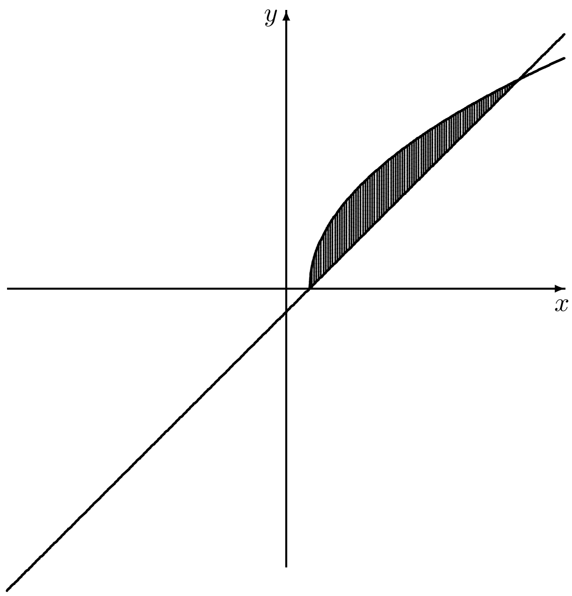

Figure 1: Question 1

[4] (b) First we need to find the points of intersection. We set

$$\begin{aligned} x - 1 &= 3\sqrt{x-1} \\ (x-1)^2 &= 9(x-1) \\ x^2 - 11x + 10 &= 0 \\ (x-10)(x-1) &= 0, \end{aligned}$$

so $x = 10$ or $x = 1$. From the graph, the top boundary curve is clearly $y = 3\sqrt{x-1}$ and the bottom boundary curve is $y = x - 1$. Thus

$$\begin{aligned} A &= \int_1^{10} \left[3\sqrt{x-1} - (x-1)\right] dx \\ &= \int_1^{10} \left[3\sqrt{x-1} - x + 1\right] dx \\ &= \left[2(x-1)^{\frac{3}{2}} - \frac{1}{2}x + x\right]_1^{10} \\ &= \frac{27}{2}. \end{aligned}$$

[4] (c) As functions of y, the line $y = x - 1$ becomes $x = y + 1$, while the semi-parabola $y = 3\sqrt{x-1}$ becomes

$$y^2 = 9(x-1) \quad \Longrightarrow \quad x = \frac{1}{9}y^2 + 1.$$

For the points of intersection, we could make use of the information we found in part (b). Substituting $x = 1$ into either curve, we get $y = 0$. Substituting $x = 10$, we get $y = 9$. Alternatively, we could solve the equation

$$\begin{aligned} \frac{1}{9}y^2 + 1 &= y + 1 \\ \frac{1}{9}y^2 - y &= 0 \\ \frac{1}{9}y(y-9) &= 0, \end{aligned}$$

so again $y = 0$ or $y = 9$. Finally, we note from the graph that $x = y + 1$ is the right boundary curve, while $x = \frac{1}{9}y^2 + 1$ is the left boundary curve. Hence

$$\begin{aligned} A &= \int_0^9 \left[(y+1) - \left(\frac{1}{9}y^2 + 1 \right) \right] dy \\ &= \int_0^9 \left[y - \frac{1}{9}y^2 \right] dy \\ &= \left[\frac{1}{2}y^2 - \frac{1}{27}y^3 \right]_0^9 \\ &= \frac{27}{2}. \end{aligned}$$

[5] 2. (a) This region is depicted in Figure 2. First we need to find the points of intersection of the two curves. We set

$$\begin{aligned} 2x^2 - 4x - 6 &= x^2 - x - 2 \\ x^2 - 3x - 4 &= 0 \\ (x+1)(x-4) &= 0, \end{aligned}$$

so $x = -1$ or $x = 4$. At $x = 0$, $x^2 - x - 2 = -2$ while $2x^2 - 4x - 6 = -6$, so

$x^2 - x - 2 \geq 2x^2 - 4x - 6$ on $[-1, 4]$. Now we have

$$\begin{aligned} A &= \int_{-1}^{4} [(x^2 - x - 2) - (2x^2 - 4x - 6)]\, dx \\ &= \int_{-1}^{4} [-x^2 + 3x + 4]\, dx \\ &= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x \right]_{-1}^{4} \\ &= \frac{125}{6}. \end{aligned}$$

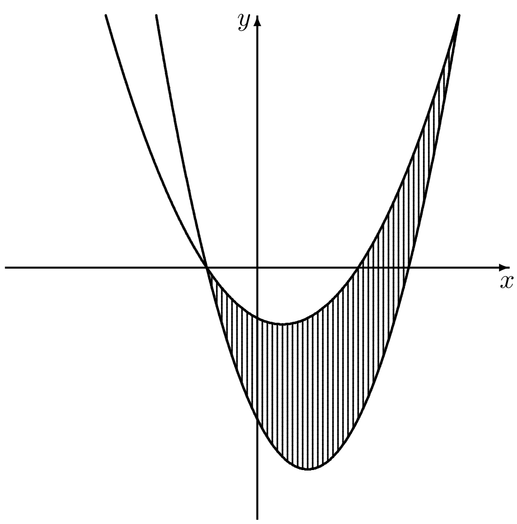

Figure 2: Question 2(a)

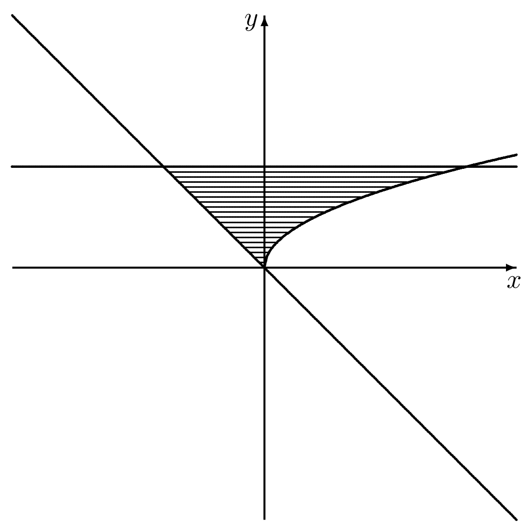

Figure 3: Question 2(b)

[5] (b) As suggested by Figure 3, this region is horizontally (but not vertically) sumple, so we have two options.

We could split the region into two vertically simple sub-regions. The first sub-region begins where the lines $y = -x$ and $y = 1$ intersect; by setting $-x = 1$, we see that this is $x = -1$. It ends where $y = -x$ intersects $y = \sqrt{x}$. If $-x = \sqrt{x}$ then $x^2 = x$ so $x^2 - x = x(x-1) = 0$. This appears to give two intersection points: $x = 0$ and $x = 1$. However, substitution back into $-x = \sqrt{x}$ shows that only $x = 0$ is a valid solution. The second sub-region must therefore start at $x = 0$ and end at the point where $y = \sqrt{x}$ intersects with $y = 1$; if $\sqrt{x} = 1$ then $x = 1^2 = 1$.

Thus we can first consider the sub-region on the interval $[-1, 0]$. Here the top boundary curve is $y = 1$ and the bottom boundary curve is $y = -x$. Hence its area is

$$A_1 = \int_{-1}^{0} [1 - (-x)]\, dx = \int_{-1}^{0} (x + 1)\, dx = \left[\frac{x^2}{2} + x \right]_{-1}^{0} = \frac{1}{2}.$$

Next we can consider the sub-region on the interval $[0,1]$. Its top boundary curve is still $y=1$ but now its bottom boundary curve is $y=\sqrt{x}$. This means that its area is

$$A_2=\int_0^1\left[1-\sqrt{x}\right]\,dx=\left[x-\frac{2}{3}x^{\frac{3}{2}}\right]_0^1=\frac{1}{3}.$$

Finally, then, the total area of the region is

$$A=A_1+A_2=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}.$$

Because the region is horizontally simple, however, it is more straightforward to work in terms of y. The function $y=\sqrt{x}$ can be written as $x=y^2$, and the function $y=-x$ becomes $x=-y$. These two curves intersect when

$$\begin{aligned}y^2&=-y\\y^2+y&=0\\y(y+1)&=0.\end{aligned}$$

This appears to suggest two intersection points $y=0$ and $y=-1$, but note that the original function $y=\sqrt{x}$ does not permit $y<0$. Thus only $y=0$ is actually an intersection point. On the y-interval $[0,1]$, y^2 lies to the right of $-y$. Hence

$$\begin{aligned}A&=\int_0^1\left[y^2-(-y)\right]dy\\&=\int_0^1\left[y^2+y\right]dy\\&=\left[\frac{1}{3}y^3+\frac{1}{2}y^2\right]_0^1\\&=\frac{5}{6}.\end{aligned}$$