MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 4 | **MATHEMATICS 1001** | Winter 2025

SOLUTIONS

[5] 1. (a) We use a regular partition with

$$\Delta x = \frac{4-(-2)}{n} = \frac{6}{n} \quad \text{and} \quad x_i^* = x_i = -2 + \frac{6i}{n}.$$

Then

$$f(x_i^*) = \left(-2 + \frac{6i}{n} - 4\right)^2 = \left(\frac{6i}{n} - 6\right)^2 = \frac{36i^2}{n^2} - \frac{72i}{n} + 36.$$

Now we can write

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{36i^2}{n^2} - \frac{72i}{n} + 36\right) \cdot \frac{6}{n} \\
&= \lim_{n\to\infty} \left[\frac{216}{n^3} \sum_{i=1}^{n} i^2 - \frac{432}{n^2} \sum_{i=1}^{n} i + \frac{216}{n} \sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty} \left[\frac{216}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{432}{n^2} \cdot \frac{n(n+1)}{2} + \frac{216}{n} \cdot n\right] \\
&= 72 - 216 + 216 \\
&= 72.
\end{aligned}$$

[5] (b) We use a regular partition of $[-1, 1]$ into n subintervals of width

$$\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}.$$

We choose the sample point

$$x_i^* = x_i = -1 + \frac{2i}{n}.$$

Hence the height of the rectangles will be

$$f\left(-1 + \frac{2i}{n}\right) = \frac{8i^3}{n^3} - \frac{8i^2}{n^2} + \frac{4i}{n}.$$

We can now write

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i^3}{n^3} - \frac{8i^2}{n^2} + \frac{4i}{n} \right) \cdot \frac{2}{n} \\ &= \lim_{n\to\infty} \left[\frac{16}{n^4} \sum_{i=1}^{n} i^3 - \frac{16}{n^3} \sum_{i=1}^{n} i^2 + \frac{8}{n^2} \sum_{i=1}^{n} i \right] \\ &= \lim_{n\to\infty} \left[\frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} - \frac{16}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{8}{n^2} \cdot \frac{n(n+1)}{2} \right] \\ &= 4 - \frac{16}{3} + 4 \\ &= \frac{8}{3}. \end{aligned}$$

[5] 2. First we need to understand the interval over which the region under $y = mx + b$ is defined. The left endpoint is $x = 0$ (since the y-axis, which is to say the line $x = 0$, is one of the boundary curves). The right endpoint will be the point at which $y = mx + b$ intersects with the x-axis, where $y = 0$. Thus we set

$$mx + b = 0 \quad \Longrightarrow \quad x = -\frac{b}{m}.$$

So we will create a regular partition of the interval $\left[0, -\frac{b}{m}\right]$ into n subintervals of width

$$\Delta x = \frac{-\frac{b}{m} - 0}{n} = -\frac{b}{mn}.$$

As usual, we set the sample point to be the right endpoint:

$$x_i^* = x_i = 0 + i\left(-\frac{b}{mn}\right) = -\frac{bi}{mn}.$$

Then the height of each rectangle is given by

$$f(x_i^*) = m\left(-\frac{bi}{mn}\right) + b = b - \frac{bi}{n}.$$

So we have

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(b - \frac{bi}{n}\right) \cdot \left(-\frac{b}{mn}\right) \\
&= \lim_{n\to\infty} \left[-\frac{b^2}{mn} \sum_{i=1}^{n} 1 + \frac{b^2}{mn^2} \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \left[-\frac{b^2}{mn} \cdot n + \frac{b^2}{mn^2} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[-\frac{b^2}{m} + \frac{b^2(n+1)}{2mn}\right] \\
&= -\frac{b^2}{m} + \frac{b^2}{2m} \\
&= -\frac{b^2}{2m}
\end{aligned}$$

as required.

[5] 3. We use a regular partition of $[0, \frac{3}{2}]$ into n subintervals of width

$$\Delta x = \frac{\frac{3}{2} - 0}{n} = \frac{3}{2n}.$$

We choose the sample point

$$x_i^* = x_i = 0 + \frac{3i}{2n} = \frac{3i}{2n}.$$

Hence the height of the rectangles will be

$$f\left(\frac{3i}{2n}\right) = 2 \cdot \frac{3i}{2n} \cdot \left(4 - \frac{9i}{2n}\right) = \frac{12i}{n} - \frac{27i^2}{2n^2}.$$

Now we have

$$\begin{aligned}
\int_0^{\frac{3}{2}} 2x(4-3x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{12i}{n} - \frac{27i^2}{2n^2}\right) \cdot \frac{3}{2n} \\
&= \lim_{n\to\infty} \left[\frac{18}{n^2} \sum_{i=1}^{n} i - \frac{81}{4n^3} \sum_{i=1}^{n} i^2\right] \\
&= \lim_{n\to\infty} \left[\frac{18}{n^2} \cdot \frac{n(n+1)}{2} - \frac{81}{4n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right] \\
&= 9 - \frac{27}{4} \\
&= \frac{9}{4}.
\end{aligned}$$