MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

SOLUTIONS

1. (a) The graph of the region can be found in Figure 1. Since the x-axis (the line $y = 0$) is one of the boundary curves, this is just a disc method problem with $R(x) = \sqrt{x-2}$. Thus

$$\begin{aligned} V &= \pi \int_2^6 \left(\sqrt{x-2}\right)^2 \, dx \\ &= \pi \int_2^6 (x-2)\, dx \\ &= \pi \left[\frac{1}{2}x^2 - 2x\right]_2^6 \\ &= \pi[(18-12) - (2-4)] \\ &= 8\pi. \end{aligned}$$

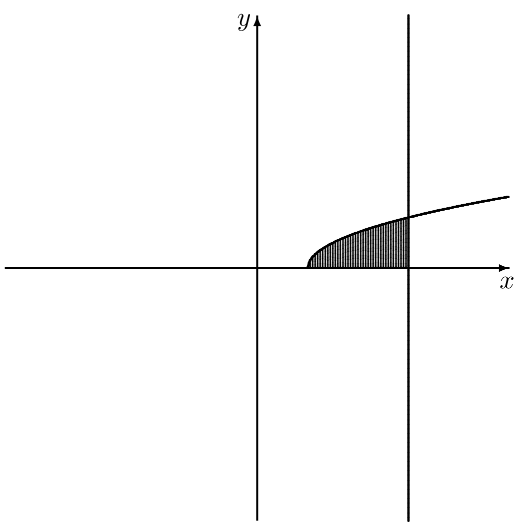

Figure 1: Question 1(a)

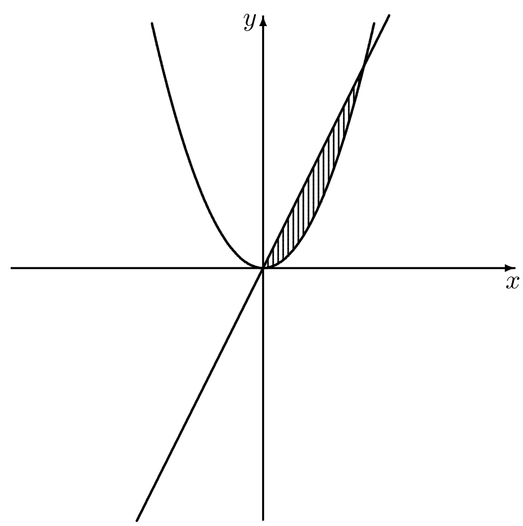

Figure 2: Question 1(b)

(b) The graph of the region can be found in Figure 2. The boundary curve $y = 2x$ is farther from the axis of revolution than the boundary curve $y = x^2$. Hence the outer radius is $R(x) = 2x$ and the inner radius is $r(x) = x^2$. Also, we find the endpoints of the interval of integration by setting

$$x^2 = 2x \quad \Longrightarrow \quad x^2 - 2x = x(x-2) = 0$$

giving $x = 0$ and $x = 2$. Thus

$$\begin{aligned} V &= \pi \int_0^2 ([2x]^2 - [x^2]^2)\,dx \\ &= \pi \int_0^2 (4x^2 - x^4)\,dx \\ &= \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 \\ &= \pi \left[\frac{32}{3} - \frac{32}{5} - 0 + 0\right] \\ &= \frac{64\pi}{15}. \end{aligned}$$

(c) The graph of the region can be found in Figure 3. The boundary curve closest to the axis is always $y = 2$ — so that $r(x) \equiv 2$ — the farther boundary curve is $y = \frac{12}{x}$ for some values of x, and $y = 3x$ for others. To determine the point where this changes, we set

$$\frac{12}{x} = 3x \quad \Longrightarrow \quad x^2 = 4 \quad \Longrightarrow \quad x = \pm 2,$$

of which only $x = 2$ is relevant to the indicated region. Also, we need the left and right endpoints of the entire interval of integration. The left endpoint is the intersection of $y = 2$ and $y = 3x$, namely $x = \frac{2}{3}$. The right endpoint is the intersection of $y = 2$ and $y = \frac{12}{x}$, which is $x = 6$. Hence on $[\frac{2}{3}, 2]$ the outer radius is $R(x) = 3x$ while on $[2, 6]$ the outer radius is $R(x) = \frac{12}{x}$. Thus we calculate

$$\begin{aligned} V &= \pi \int_{\frac{2}{3}}^2 ([3x]^2 - [2]^2)\,dx + \pi \int_2^6 \left(\left[\frac{12}{x}\right]^2 - [2]^2\right) dx \\ &= \pi \int_{\frac{2}{3}}^2 (9x^2 - 4)\,dx + \pi \int_2^6 \left(\frac{144}{x^2} - 4\right) dx \\ &= \pi \left[\left[3x^3 - 4x\right]\right]_{\frac{2}{3}}^2 + \pi \left[-\frac{144}{x} - 4x\right]_2^6 \\ &= \pi \left[(24 - 8) - \left(\frac{8}{9} - \frac{8}{3}\right)\right] + \pi\left[(-24 - 24) - (-72 - 8)\right] \\ &= \frac{160\pi}{9} + 32\pi \\ &= \frac{448\pi}{9}. \end{aligned}$$

2. The plane shape which will generate the desired cone by revolving it around the x-axis is a right triangle with base b and height a. We can form such a plane shape by considering the

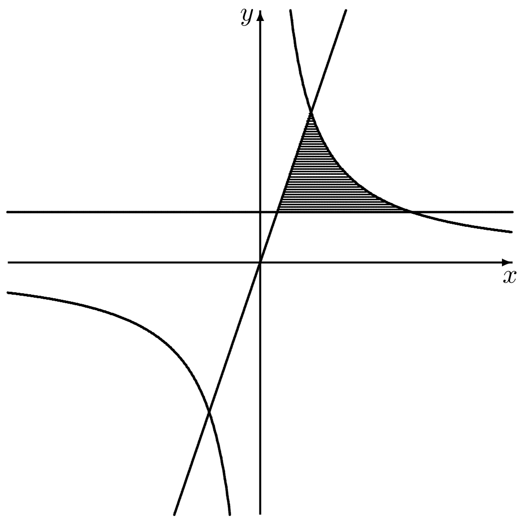

Figure 3: Question 1(c)

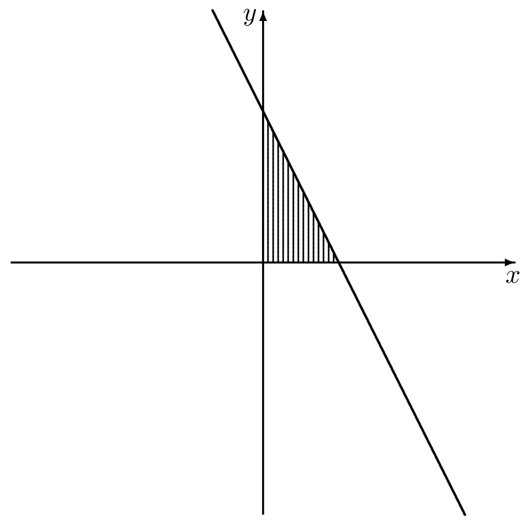

Figure 4: Question 2

region bounded by the curves $y = -\frac{a}{b}x + a$, $y = 0$ and $x = 0$, as depicted in Figure 4. The result is a disc method problem with outer radius $R(x) = -\frac{a}{b}x + a$, so

$$\begin{aligned} V &= \pi \int_0^b \left[-\frac{a}{b}x + a\right]^2 \, dx \\ &= \pi \int_0^b \left[\frac{a^2}{b^2}x^2 - \frac{2a^2}{b}x + a^2\right] \, dx \\ &= \pi \left[\frac{a^2}{3b^2}x^3 - \frac{a^2}{b}x^2 + a^2x\right]_0^b \\ &= \frac{\pi}{3}a^2b. \end{aligned}$$