

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.4

Math 1001 Worksheet

WINTER 2025

SOLUTIONS

1. (a) Note that $x(t)$ is negatively impacted by the interactions between the two populations, as indicated by the term $-2xy$ in the first differential equation, while $y(t)$ is positively impacted, as indicated by the term $+3xy$ in the second differential equation. Hence we can conclude that $x(t)$ is the prey and $y(t)$ is the predator.

- (b) If $y = 0$ then $x(t)$ is described by the differential equation

$$\frac{dx}{dt} = 5x,$$

which indicates exponential growth. Hence no other factor inhibits the growth of the prey population.

- (c) First we set $5x - 2xy = 0$ so $x(5 - 2y) = 0$ and either $\bar{x} = 0$ or $\bar{y} = \frac{5}{2}$. Now we consider $-y + 3xy = 0$. If $\bar{x} = 0$ then this becomes $-y = 0$ so $\bar{y} = 0$. If $\bar{y} = \frac{5}{2}$ then it becomes

$$-\frac{5}{2} + 3x \cdot \frac{5}{2} = 0 \implies \bar{x} = \frac{1}{3}.$$

Hence the equilibrium points are $(0, 0)$ which indicates the extinction of both populations, and $(\frac{1}{3}, \frac{5}{2})$ which indicates that a prey population of $x(t) = \frac{1}{3}$ perfectly supports a predator population of $y(t) = \frac{5}{2}$.

2. (a) Note that $x(t)$ is positively impacted by the interactions between the two populations, as indicated by the term $+8xy$ in the first differential equation, while $y(t)$ is negatively impacted, as indicated by the term $-xy$ in the second differential equation. Hence we can conclude that $x(t)$ is the predator and $y(t)$ is the prey.

- (b) If $x = 0$ then $y(t)$ is described by the differential equation

$$\frac{dy}{dt} = 3y - 6y^2 = 3y(1 - 2y),$$

which is the form of the logistic model. Hence the prey population is inhibited by environmental factors such as food availability, in addition to the actions of the predator population.

- (c) First we set $-4x + 8xy = 0$ so $x(-4 + 8y) = 0$ and either $\bar{x} = 0$ or $\bar{y} = \frac{1}{2}$. Now we consider $3y - 6y^2 - xy = 0$. If $\bar{x} = 0$ then this becomes

$$3y - 6y^2 = 0 \implies 3y(1 - 2y) = 0$$

so either $\bar{y} = 0$ or $\bar{y} = \frac{1}{2}$. If $\bar{y} = \frac{1}{2}$ then it becomes

$$3 \cdot \frac{1}{2} - 6 \cdot \frac{1}{4} - x \cdot \frac{1}{2} = 0 \implies \frac{1}{2}x = 0$$

so $\bar{x} = 0$, as we've already determined. Hence the equilibrium points are $(0, 0)$ which indicates the extinction of both populations, and $(0, \frac{1}{2})$ which indicates the extinction of the predator population and the survival of the prey population.

3. (a) Since their interactions positively impact both populations, as indicated by the term $+8xy$ in the first differential equation and the term $+xy$ in the second differential equation, this model describes **co-operation**.
- (b) First we set $-4x + 4xy = 0$ so $x(-4 + 4y) = 0$ and either $\bar{x} = 0$ or $\bar{y} = 1$. Now we consider $3y - 6y^2 + xy = 0$. If $\bar{x} = 0$ then this becomes

$$3y - 6y^2 = 0 \implies 3y(1 - 2y) = 0$$

so either $\bar{y} = 0$ or $\bar{y} = \frac{1}{2}$. If $\bar{y} = 1$ then it becomes

$$3 \cdot 1 - 6 \cdot 1 + x \cdot 1 = 0$$

so $\bar{x} = 3$. Hence the equilibrium points are $(0, 0)$ which indicates the extinction of both populations, $(0, \frac{1}{2})$ which indicates the extinction of $x(t)$ and the survival of $y(t)$, and $(3, 1)$ which indicates the survival of both populations.

4. (a) Since their interactions negatively impact both populations, as indicated by the term $-5xy$ in the first differential equation and the term $-4xy$ in the second differential equation, this model describes **competition**.
- (b) First we set $6x - 2x^2 - 5xy = 0$ so $x(6 - 2x - 5y) = 0$ and either $\bar{x} = 0$ or $\bar{x} = 3 - \frac{5}{2}y$. Now we consider $8y - 5y^2 - 4xy = 0$. If $\bar{x} = 0$ then this becomes

$$8y - 5y^2 = 0 \implies y(8 - 5y) = 0$$

so either $\bar{y} = 0$ or $\bar{y} = \frac{8}{5}$. If $\bar{x} = 3 - \frac{5}{2}y$ then it becomes

$$8y - 5y^2 - 4\left(3 - \frac{5}{2}y\right)y = 0$$

$$8y - 5y^2 - 12y + 10y^2 = 0$$

$$5y^2 - 4y = 0$$

$$y(5y - 4) = 0$$

so either $\bar{y} = 0$, in which case $\bar{x} = 3$, or $\bar{y} = \frac{4}{5}$, in which case $\bar{x} = 1$. Hence we have four equilibrium points. First, $(0, 0)$ indicates the extinction of both populations. Second, $(0, \frac{8}{5})$ indicates the extinction of $x(t)$ and the survival of $y(t)$. Third, $(3, 0)$ indicates the extinction of $y(t)$ and the survival of $x(t)$. Finally, $(1, \frac{4}{5})$ indicates the survival of both populations.