# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

## DEPARTMENT OF MATHEMATICS AND STATISTICS

| Section 4.1 | **Math 1001 Worksheet** | Winter 2024 |
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## SOLUTIONS

1. (a) Note that

$$\frac{dy}{dt} = 1 \quad \text{and} \quad \frac{d^2y}{dt^2} = 0,$$

so

$$t^2\frac{d^2y}{dt^2} + y = t^2 \cdot 0 + t = t$$

and

$$t\frac{dy}{dt} = t \cdot 1 = t.$$

Since these expressions are equal, the differential equation is satisfied, and hence $y = t$ is a solution of the given equation.

(b) Note that

$$\frac{dy}{dt} = \frac{1}{t} \quad \text{and} \quad \frac{d^2y}{dt^2} = -\frac{1}{t^2},$$

so

$$t^2\frac{d^2y}{dt^2} + y = t^2\left(-\frac{1}{t^2}\right) + \ln(t) = \ln(t) - 1$$

and

$$t\frac{dy}{dt} = t\left(\frac{1}{t}\right) = 1.$$

Since these expressions are not equal, the differential equation is not satisfied, and hence $y = \ln(t)$ is not a solution of the given equation.

(c) Note that

$$\frac{dy}{dt} = \ln(t) + 1 \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{1}{t},$$

so

$$t^2\frac{d^2y}{dt^2} + y = t^2\left(\frac{1}{t}\right) + t\ln(t) = t + t\ln(t)$$

and

$$t\frac{dy}{dt} = t[\ln(t) + 1] = t\ln(t) + t.$$

Since these expressions are equal, the differential equation is again satisfied, and hence $y = t\ln(t)$ is also a solution of the given equation.

2. (a) We have

$$\begin{aligned} \frac{dy}{dt} &= 9 - \sqrt{t} \\ y(t) &= \int \left(9 - \sqrt{t}\right)\, dt \\ &= 9t - \frac{2}{3}t^{\frac{3}{2}} + C. \end{aligned}$$

This is the general solution, so to find the particular solution we will use the fact that $y = 4$ when $t = 0$. Thus

$$y(0) = 0 - 0 + C = C = 4.$$

The particular solution is therefore

$$\boxed{y(t) = 9t - \frac{2}{3}t^{\frac{3}{2}} + 4.}$$

(b) We have

$$\begin{aligned} \cos^2(t)\frac{dy}{dt} &= 1 - \cos(t) \\ \frac{dy}{dt} &= \sec^2(t) - \sec(t) \\ y(t) &= \int [\sec^2(t) - \sec(t)]\, dt \\ &= \tan(t) - \ln|\sec(t) + \tan(t)| + C. \end{aligned}$$

Since $y(0) = 0$, we obtain

$$y(0) = 0 - \ln|1 + 0| + C = C = 0,$$

so the particular solution is

$$\boxed{y(t) = \tan(t) - \ln|\sec(t) + \tan(t)|.}$$

(c) We have

$$\begin{aligned} f'(t) &= \frac{\ln(t)}{t^2} \\ f(t) &= \int \frac{\ln(t)}{t^2}\, dt. \end{aligned}$$

We use integration by parts with $w = \ln(t)$ so $dw = \frac{1}{t}\, dt$ and $dv = \frac{1}{t^2}\, dt$ so $v = -\frac{1}{t}$. The integral becomes

$$\begin{aligned} f(t) &= -\frac{\ln(t)}{t} + \int \frac{1}{t^2}\, dt \\ &= -\frac{\ln(t)}{t} - \frac{1}{t} + C. \end{aligned}$$

Since $f(1) = 2$, we get

$$f(1) = -\frac{\ln(1)}{1} - 1 + C = C - 1 = 2$$

and so $C = 3$. Hence the particular solution is

$$\boxed{f(t) = -\frac{\ln(t)}{t} - \frac{1}{t} + 3.}$$

(d) We rewrite and integrate:

$$\begin{aligned} f''(t) &= 4t^{-2} \\ \int f''(t)\,dt &= 4\int t^{-2}\,dt \\ f'(t) &= 4\left[\frac{1}{-1}t^{-1}\right] + C = -\frac{4}{t} + C. \end{aligned}$$

This gives $f'(1) = -4 + C = 0$ and so $C = 4$. Now we integrate a second time:

$$\begin{aligned} \int f'(t)\,dt &= \int (-4t^{-1} + 4)\,dt \\ f(t) &= -4\ln|t| + 4t + C \end{aligned}$$

yielding $f(-1) = -4\ln|-1| + 4(-1) + C = -4 + C$. Then we can set $-4 + C = 3$ to get $C = 7$, and the particular solution is

$$\boxed{f(t) = -4\ln|t| + 4t + 7.}$$

(e) Integrating twice gives

$$\begin{aligned} \int f''(t)\,dt &= \int (3t - 3)\,dt \\ f'(t) &= \frac{3}{2}t^2 - 3t + C \\ \int f'(t)\,dt &= \int \left(\frac{3}{2}t^2 - 3t + C\right)\,dt \\ f(t) &= \frac{3}{2}\left[\frac{1}{3}t^3\right] - 3\left[\frac{1}{2}t^2\right] + Ct + D = \frac{1}{2}t^3 - \frac{3}{2}t^2 + Ct + D, \end{aligned}$$

where both $C$ and $D$ are arbitrary constants. Using the first initial condition, we have that $f(0) = D = -5$. Using the other condition, we get $f(2) = \frac{1}{2}(8) - \frac{3}{2}(4) + C(2) - 5 = 4 - 6 + 2C - 5 = 2C - 7$. Then we set $2C - 7 = -7$ to get $C = 0$. Hence the particular solution is

$$\boxed{f(t) = \frac{1}{2}t^3 - \frac{3}{2}t^2 - 5.}$$

3. Integrating gives

$$\int f'(x)\,dx = \int 9x^2\,dx$$
$$f(x) = 9\left[\frac{1}{3}x^3\right] + C = 3x^3 + C.$$

We want the line $y = 36x$ to be tangent to the graph $y = f(x)$, that is, to $y = 3x^3 + C$. This means that the two curves must meet at a point where their slopes are equal. But the slope of $y = 36x$ is always $y' = 36$, so we solve $f'(x) = 36$, giving

$$9x^2 = 36 \quad\Longrightarrow\quad x^2 = 4 \quad\Longrightarrow\quad x = \pm 2.$$

In the first case, from the equation of the line we have $y = 36(2) = 72$ so then

$$3(2)^3 + C = 72 \quad\Longrightarrow\quad 24 + C = 72 \quad\Longrightarrow\quad C = 48.$$

In the second case, we have $y = 36(-2) = -72$ and thus

$$3(-2)^3 + C = -72 \quad\Longrightarrow\quad -24 + C = -72 \quad\Longrightarrow\quad C = -48.$$

Hence the two such functions are

$$f(x) = 3x^3 + 48 \quad \text{and} \quad f(x) = 3x^3 - 48.$$

4. (a) The acceleration function is simply $a(t) = -9.8$, so integrating twice gives us both the velocity and position functions:

$$\int a(t)\,dt = \int (-9.8)\,dt$$
$$v(t) = -9.8t + C$$
$$\int v(t)\,dt = \int (-9.8t + C)\,dt$$
$$s(t) = -9.8\left[\frac{1}{2}t^2\right] + Ct + D = -4.9t^2 + Ct + D.$$

We are told that the rocket is launched from the ground, which implies that $s(0) = 0$, and so $D = 0$. Now let the time at which the rocket reaches its maximum height be $T$; then $v(T) = 0$ and we have that $-9.8T + C = 0$ so $T = \frac{C}{9.8}$. We want $s(T) = 4410$, so then

$$s(T) = -4.9T^2 + CT$$
$$4410 = -4.9\left(\frac{C}{9.8}\right)^2 + C\left(\frac{C}{9.8}\right)$$
$$4410 = \frac{C^2}{19.6}$$
$$C^2 = 86436$$
$$C = \pm 294.$$

Finally, we have the initial velocity $v(0) = C = \pm 294$. Since the rocket is launched upward, we can accept only the positive answer; hence the initial velocity must be 294 metres per second.

(b) From the above, the rocket reaches its maximum height when $T = \frac{C}{9.8} = \frac{294}{9.8} = 30$, that is, after 30 seconds.

(c) The particular solution is $s(t) = -4.9t^2 + 294t$, so

$$s(10) = -4.9(100) + 294(10) = 2450.$$

The rocket is 2450 metres high after 10 seconds.