

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 3.3

Math 1001 Worksheet

WINTER 2024

SOLUTIONS

1. (a) We let $x = 6 \sec(\theta)$ so $dx = 6 \sec(\theta) \tan(\theta) d\theta$ and $\sqrt{x^2 - 36} = 6 \tan(\theta)$. The integral becomes

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - 36}} dx &= \int \frac{6 \sec(\theta) \tan(\theta)}{6 \tan(\theta)} d\theta \\ &= \int \sec(\theta) d\theta \\ &= \ln|\sec(\theta) + \tan(\theta)| + C \\ &= \ln \left| \frac{x}{6} + \frac{\sqrt{x^2 - 36}}{6} \right| + C. \end{aligned}$$

- (b) We let $x = \sqrt{2} \sin(\theta)$, so $dx = \sqrt{2} \cos(\theta) d\theta$, $\sqrt{2 - x^2} = \sqrt{2} \cos(\theta)$ and so $(2 - x^2)^{\frac{3}{2}} = 2\sqrt{2} \cos^3(\theta)$. The integral becomes

$$\begin{aligned} \int \frac{1}{(2 - x^2)^{\frac{3}{2}}} dx &= \int \frac{\sqrt{2} \cos(\theta)}{2\sqrt{2} \cos^3(\theta)} d\theta \\ &= \frac{1}{2} \int \sec^2(\theta) d\theta \\ &= \frac{1}{2} \tan(\theta) + C. \end{aligned}$$

Now note that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{1}{\sqrt{2}}x}{\frac{1}{\sqrt{2}}\sqrt{2 - x^2}} = \frac{x}{\sqrt{2 - x^2}}.$$

Thus

$$\int \frac{1}{(2 - x^2)^{\frac{3}{2}}} dx = \frac{x}{2\sqrt{2 - x^2}} + C.$$

- (c) We let $x = 4 \sin(\theta)$ so $dx = 4 \cos(\theta) d\theta$ and $\sqrt{16 - x^2} = 4 \cos(\theta)$. Also note that $x = 0$ implies $\sin(\theta) = 0$ and so $\theta = 0$, while $x = 2$ implies $\sin(\theta) = \frac{1}{2}$ and so $\theta = \frac{\pi}{6}$. The

integral therefore becomes

$$\begin{aligned}
 \int_0^2 \sqrt{16-x^2} dx &= \int_0^{\frac{\pi}{6}} [4 \cos(\theta)] \cdot 4 \cos(\theta) d\theta \\
 &= 16 \int_0^{\frac{\pi}{6}} \cos^2(\theta) d\theta \\
 &= 16 \int_0^{\frac{\pi}{6}} \left[\frac{1 + \cos(2\theta)}{2} \right] d\theta \\
 &= 8 \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{\frac{\pi}{6}} \\
 &= 8 \left[\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right] \\
 &= \frac{4\pi}{3} + 2\sqrt{3}.
 \end{aligned}$$

2. Note that

$$\sqrt{e^{6x} - 4} = \sqrt{(e^{3x})^2 - 4}.$$

We let $u = e^{3x}$, so $du = 3e^{3x} dx = 3u dx$ and thus $\frac{1}{3u} du = dx$. Now we have

$$\int \sqrt{e^{6x} - 4} dx = \frac{1}{3} \int \frac{\sqrt{u^2 - 4}}{u} du.$$

Next we let $u = 2 \sec(\theta)$ so $du = 2 \sec(\theta) \tan(\theta) d\theta$ and $\sqrt{u^2 - 4} = 2 \tan(\theta)$. The integral becomes

$$\begin{aligned}
 \int \sqrt{e^{6x} - 4} dx &= \frac{1}{3} \int \frac{2 \tan(\theta)}{2 \sec(\theta)} \cdot 2 \sec(\theta) \tan(\theta) d\theta \\
 &= \frac{2}{3} \int \tan^2(\theta) d\theta \\
 &= \frac{2}{3} \int [\sec^2(\theta) - 1] d\theta \\
 &= \frac{2}{3} [\tan(\theta) - \theta] + C \\
 &= \frac{2}{3} \left[\frac{1}{2} \sqrt{u^2 - 4} - \operatorname{arcsec} \left(\frac{u}{2} \right) \right] + C \\
 &= \frac{2}{3} \left[\frac{1}{2} \sqrt{e^{6x} - 4} - \operatorname{arcsec} \left(\frac{e^{3x}}{2} \right) \right] + C \\
 &= \frac{1}{3} \sqrt{e^{6x} - 4} - \frac{2}{3} \operatorname{arcsec} \left(\frac{e^{3x}}{2} \right) + C.
 \end{aligned}$$

3. We complete the square:

$$\begin{aligned} 4x - x^2 &= -(x^2 - 4x) \\ &= -(x^2 - 4x + 4) + 4 \\ &= -(x - 2)^2 + 4 \\ &= 4 - (x - 2)^2. \end{aligned}$$

Thus we can rewrite the integral as

$$\int \frac{x^2}{\sqrt{4x - x^2}} dx = \int \frac{x^2}{\sqrt{4 - (x - 2)^2}} dx.$$

Now let $u = x - 2$ so $du = dx$ and $x = u + 2$. Then we have

$$\int \frac{x^2}{\sqrt{4 - (x - 2)^2}} dx = \int \frac{(u + 2)^2}{\sqrt{4 - u^2}} du = \int \frac{u^2 + 4u + 4}{\sqrt{4 - u^2}} du.$$

Now we let $u = 2 \sin(\theta)$ so $du = 2 \cos(\theta) d\theta$ and $\sqrt{4 - u^2} = 2 \cos(\theta)$. The integral becomes

$$\begin{aligned} &\int \frac{x^2}{\sqrt{4x - x^2}} dx \\ &= \int \frac{4 + 8 \sin(\theta) + 4 \sin^2(\theta)}{2 \cos(\theta)} \cdot 2 \cos(\theta) d\theta \\ &= \int [4 + 8 \sin(\theta) + 4 \sin^2(\theta)] d\theta \\ &= \int [4 + 8 \sin(\theta) + 2(1 - \cos(2\theta))] d\theta \\ &= \int [6 + 8 \sin(\theta) - 2 \cos(2\theta)] d\theta \\ &= 6\theta - 8 \cos(\theta) - \sin(2\theta) + C \\ &= 6\theta - 8 \cos(\theta) - 2 \sin(\theta) \cos(\theta) + C \\ &= 6 \arcsin\left(\frac{u}{2}\right) - 8 \cdot \frac{\sqrt{4 - u^2}}{2} - 2 \cdot \frac{u}{2} \cdot \frac{\sqrt{4 - u^2}}{2} + C \\ &= 6 \arcsin\left(\frac{u}{2}\right) - 4\sqrt{4 - u^2} - \frac{1}{2}u\sqrt{4 - u^2} + C \\ &= 6 \arcsin\left(\frac{x - 2}{2}\right) - 4\sqrt{4 - (x - 2)^2} - \frac{1}{2}(x - 2)\sqrt{4 - (x - 2)^2} + C \\ &= 6 \arcsin\left(\frac{x - 2}{2}\right) - 3\sqrt{4 - (x - 2)^2} - \frac{1}{2}x\sqrt{4 - (x - 2)^2} + C. \end{aligned}$$

4. (a) We use u -substitution, with $u = x^2 + 4$ so $\frac{1}{2} du = x dx$. Then

$$\begin{aligned} \int \frac{x}{\sqrt{x^2 + 4}} dx &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \\ &= \frac{1}{2} [2\sqrt{u}] + C \\ &= \sqrt{x^2 + 4} + C. \end{aligned}$$

(b) We use trigonometric substitution, with $x = 2 \sin(\theta)$. Then $dx = 2 \cos(\theta) d\theta$ and $\sqrt{4 - x^2} = 2 \cos(\theta)$. The integral becomes

$$\begin{aligned} \int \frac{\sqrt{4 - x^2}}{x} dx &= \int \frac{2 \cos(\theta)}{2 \sin(\theta)} \cdot 2 \cos(\theta) d\theta \\ &= 2 \int \frac{\cos^2(\theta)}{\sin(\theta)} d\theta \\ &= 2 \int \frac{1 - \sin^2(\theta)}{\sin(\theta)} d\theta \\ &= 2 \int [\csc(\theta) - \sin(\theta)] d\theta \\ &= -2 \ln|\csc(\theta) + \cot(\theta)| + 2 \cos(\theta) + C. \end{aligned}$$

We already know that $\sin(\theta) = \frac{1}{2}x$, so $\csc(\theta) = \frac{2}{x}$. Furthermore, $\cos(\theta) = \frac{1}{2}\sqrt{4 - x^2}$. Finally,

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\frac{1}{2}\sqrt{4 - x^2}}{\frac{1}{2}x} = \frac{\sqrt{4 - x^2}}{x}.$$

Hence

$$\begin{aligned} \int \frac{\sqrt{4 - x^2}}{x} dx &= -2 \ln \left| \frac{2}{x} + \frac{\sqrt{4 - x^2}}{x} \right| + 2 \left(\frac{1}{2} \sqrt{4 - x^2} \right) + C \\ &= -2 \ln \left| \frac{\sqrt{4 - x^2} + 2}{x} \right| + \sqrt{4 - x^2} + C. \end{aligned}$$

(c) We complete the square:

$$\begin{aligned} 4x^2 - 12x + 13 &= 4 \left[x^2 - 3x + \frac{13}{4} \right] \\ &= 4 \left[\left(x^2 - 3x + \frac{9}{4} \right) + 1 \right] \\ &= 4 \left(x - \frac{3}{2} \right)^2 + 4 \\ &= (2x - 3)^2 + 4. \end{aligned}$$

Thus

$$\int \frac{1}{4x^2 - 12x + 13} dx = \int \frac{1}{(2x - 3)^2 + 4} dx.$$

Let $u = 2x - 3$ so $\frac{1}{2} du = dx$. The integral becomes

$$\begin{aligned} \frac{1}{2} \int \frac{1}{u^2 + 4} du &= \frac{1}{2} \cdot \frac{1}{2} \arctan\left(\frac{u}{2}\right) + C \\ &= \frac{1}{4} \arctan\left(\frac{2x - 3}{2}\right) + C. \end{aligned}$$

(d) We can factor the denominator:

$$4x^2 - 4x - 3 = (2x + 1)(2x - 3).$$

Thus an appropriate partial fraction decomposition is

$$\begin{aligned} \frac{1}{4x^2 - 4x - 3} &= \frac{A}{2x + 1} + \frac{B}{2x - 3} \\ 1 &= A(2x - 3) + B(2x + 1). \end{aligned}$$

When $x = -\frac{1}{2}$, we get $1 = -4A$ so $A = -\frac{1}{4}$. When $x = \frac{3}{2}$, we get $1 = 4B$ so $B = \frac{1}{4}$. Thus

$$\begin{aligned} \frac{1}{4x^2 - 4x - 3} dx &= \int \left(\frac{-\frac{1}{4}}{2x + 1} + \frac{\frac{1}{4}}{2x - 3} \right) dx \\ &= -\frac{1}{4} \cdot \frac{1}{2} \ln|2x + 1| + \frac{1}{4} \cdot \frac{1}{2} \ln|2x - 3| + C \\ &= \frac{1}{8} \ln|2x - 3| - \frac{1}{8} \ln|2x + 1| + C. \end{aligned}$$

(e) We use u -substitution, with $u = \ln(x)$ so $du = \frac{1}{x} dx$. The integral becomes

$$\begin{aligned} \int \frac{1}{x[\ln(x)]^2} dx &= \int \frac{1}{u^2} du \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{\ln(x)} + C. \end{aligned}$$

(f) We use integration by parts. Let $w = [\ln(x)]^2$ so $dw = \frac{2\ln(x)}{x} dx$. Let $dv = x dx$ so $v = \frac{1}{2}x^2$. Then

$$\int x[\ln(x)]^2 dx = \frac{1}{2}x^2[\ln(x)]^2 - \int x \ln(x) dx.$$

We need to use integration by parts a second time. Let $w = \ln(x)$ so $dw = \frac{1}{x} dx$, and let $dv = x dx$ so $v = \frac{1}{2}x^2$. Now

$$\begin{aligned}\int x[\ln(x)]^2 dx &= \frac{1}{2}x^2[\ln(x)]^2 - \left[\frac{1}{2}x^2 \ln(x) - \frac{1}{2} \int x dx \right] \\ &= \frac{1}{2}x^2[\ln(x)]^2 - \frac{1}{2}x^2 \ln(x) + \frac{1}{4}x^2 + C.\end{aligned}$$

- (g) We use a trig integration strategy. Since the powers of $\sin(x)$ and $\cos(x)$ are both odd, we can use one of two approaches.

First, we could factor out a $\cos(x)$ and then use the identity

$$\cos^2(x) = 1 - \sin^2(x)$$

to obtain

$$\begin{aligned}\int \sin^3(x) \cos^3(x) dx &= \int \sin^3(x) \cos^2(x) \cdot \cos(x) dx \\ &= \int \sin^3(x)[1 - \sin^2(x)] \cos(x) dx.\end{aligned}$$

Now let $u = \sin(x)$ so $du = \cos(x) dx$. The integral becomes

$$\begin{aligned}\int \sin^3(x) \cos^3(x) dx &= \int u^3[1 - u^2] du \\ &= \int [u^3 - u^5] du \\ &= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C \\ &= \frac{1}{4} \sin^4(x) - \frac{1}{6} \sin^6(x) + C.\end{aligned}$$

Alternatively, we could factor out a $\sin(x)$ and then use the identity

$$\sin^2(x) = 1 - \cos^2(x)$$

to obtain

$$\begin{aligned}\int \sin^3(x) \cos^3(x) dx &= \int \sin^2(x) \cos^3(x) \cdot \sin(x) dx \\ &= \int [1 - \cos^2(x)] \cos^3(x) \cdot \sin(x) dx.\end{aligned}$$

Now let $u = \cos(x)$ so $du = -\sin(x) dx$ and $-du = \sin(x) dx$. The integral becomes

$$\begin{aligned} \int \sin^3(x) \cos^3(x) dx &= - \int [1 - u^2] u^3 du \\ &= - \int [u^3 - u^5] du \\ &= \frac{1}{6} u^6 - \frac{1}{4} u^4 + C \\ &= \frac{1}{6} \cos^6(x) - \frac{1}{4} \cos^4(x) + C. \end{aligned}$$

(h) Here we simply have to realise that

$$\sin^3(x) \csc^3(x) = \sin^3(x) \cdot \frac{1}{\sin^3(x)} = 1,$$

so

$$\int \sin^3(x) \csc^3(x) dx = \int dx = x + C.$$

(i) This is trickier. First note that this is not an elementary integral, and we can rule out most techniques: u -substitution, trig substitution, partial fractions, trig integrals. That pretty much means that we must be able to use integration by parts. However, as written, parts doesn't get us anywhere: there's no choice of w and dv that will provide a manageable integral. So, instead, let's rewrite the integral as

$$\int x[\sec^2(x) - 1] dx = \int x \sec^2(x) dx - \int x dx = \int x \sec^2(x) dx - \frac{1}{2} x^2.$$

For the remaining integral, parts is a viable option. Let $w = x$ so $dw = dx$. Let $dv = \sec^2(x) dx$ so $v = \tan(x)$. Then

$$\begin{aligned} \int x \sec^2(x) dx &= x \tan(x) - \int \tan(x) dx \\ &= x \tan(x) + \ln|\cos(x)| + C. \end{aligned}$$

Finally, then,

$$\int x \tan^2(x) dx = x \tan(x) + \ln|\cos(x)| - \frac{1}{2} x^2 + C.$$