MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 3.2 | **Math 1001 Worksheet** | WINTER 2024

SOLUTIONS

1. (a) We extract a factor of $\cos(6x)$ and let $u = \sin(6x)$ so $\frac{1}{6}\,du = \cos(6x)\,dx$. Note that $x = \frac{\pi}{9}$ implies $u = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, and $x = 0$ implies $u = 0$. Hence the integral becomes

$$\begin{aligned}
\int_0^{\frac{\pi}{9}} \sin^2(6x)\cos^3(6x)\,dx &= \int_0^{\frac{\pi}{9}} \sin^2(6x)\cos^2(6x)\cos(6x)\,dx \\
&= \int_0^{\frac{\pi}{9}} \sin^2(6x)[1-\sin^2(6x)]\cos(6x)\,dx \\
&= \frac{1}{6}\int_0^{\frac{\sqrt{3}}{2}} u^2[1-u^2]\,du \\
&= \frac{1}{6}\int_0^{\frac{\sqrt{3}}{2}} [u^2-u^4]\,du \\
&= \frac{1}{6}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^{\frac{\sqrt{3}}{2}} \\
&= \frac{11\sqrt{3}}{960}.
\end{aligned}$$

(b) We extract a factor of $\sin(x)$ and let $u = \cos(x)$ so $-\,du = \sin(x)\,dx$. The integral becomes

$$\begin{aligned}
\int \sin^3(x)\cos^8(x)\,dx &= \int \sin^2(x)\cos^8(x)\sin(x)\,dx \\
&= \int [1-\cos^2(x)]\cos^8(x)\sin(x)\,dx \\
&= -\int [1-u^2]u^8\,du \\
&= \int [u^{10}-u^8]\,du \\
&= \frac{1}{11}u^{11} - \frac{1}{9}u^9 + C \\
&= \frac{1}{11}\cos^{11}(x) - \frac{1}{9}\cos^9(x) + C.
\end{aligned}$$

(c) We extract a factor of $\cos(x)$ and let $u = \sin(x)$ so $du = \cos(x)\,dx$. The integral becomes

$$\begin{aligned}\int \sin^2(x)\cos^5(x)\,dx &= \int \sin^2(x)\cos^4(x)\cos(x)\,dx \\ &= \int \sin^2(x)[1-\sin^2(x)]^2\cos(x)\,dx \\ &= \int u^2[1-u^2]^2\,du \\ &= \int [u^2 - 2u^4 + u^6]\,du \\ &= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C \\ &= \frac{1}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + \frac{1}{7}\sin^7(x) + C.\end{aligned}$$

(d) First let $u = \ln(x)$ so $du = \frac{1}{x}\,dx$, so the integral becomes

$$\int \frac{\cos^3(\ln(x))}{x}\,dx = \int \cos^3(u)\,du = \int \cos^2(u)\cos(u)\,du = \int [1-\sin^2(u)]\cos(u)\,du.$$

Now let $v = \sin(u)$ so $dv = \cos(u)\,du$ and we get

$$\begin{aligned}\int \frac{\cos^3(\ln(x))}{x}\,dx &= \int [1-v^2]\,dv \\ &= v - \frac{1}{3}v^3 + C \\ &= \sin(u) - \frac{1}{3}\sin^3(u) + C \\ &= \sin(\ln(x)) - \frac{1}{3}\sin^3(\ln(x)) + C.\end{aligned}$$

(e) We begin by using integration by parts. Let $w = x$ so $dw = dx$, and let $dv = \sin^2(x)\,dx$. To find v, then, we use the half-angle formula and integrate:

$$v = \int \sin^2(x)\,dx = \frac{1}{2}\int [1-\cos(2x)]\,dx = \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] = \frac{1}{2}x - \frac{1}{4}\sin(2x)$$

where, as usual, we have omitted the constant of integration. Hence

$$\begin{aligned}\int x\sin^2(x)\,dx &= \frac{1}{2}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{2}\int \left[x - \frac{1}{2}\sin(2x)\right]\,dx \\ &= \frac{1}{2}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{4}x^2 - \frac{1}{8}\cos(2x) + C \\ &= \frac{1}{4}x^2 - \frac{1}{8}\cos(2x) - \frac{1}{4}x\sin(2x) + C.\end{aligned}$$

(f) We simply have

$$\begin{aligned}\int \frac{1-\tan^2(x)}{\sec^2(x)}\,dx &= \int \frac{1-[\sec^2(x)-1]}{\sec^2(x)}\,dx\\ &= \int \frac{2-\sec^2(x)}{\sec^2(x)}\,dx\\ &= \int [2\cos^2(x)-1]\,dx\\ &= \int [1+\cos(2x)-1]\,dx\\ &= \int \cos(2x)\,dx\\ &= \frac{1}{2}\sin(2x)+C.\end{aligned}$$

2. (a) We extract a factor of $\sec(x)\tan(x)$ and let $u = \sec(x)$ so $du = \sec(x)\tan(x)\,dx$. The integral becomes

$$\begin{aligned}\int \tan^5(x)\sec^5(x)\,dx &= \int \tan^4(x)\sec^4(x)\sec(x)\tan(x)\,dx\\ &= \int [\sec^2(x)-1]^2\sec^4(x)\sec(x)\tan(x)\,dx\\ &= \int [u^2-1]^2u^4\,du\\ &= \int [u^8-2u^6+u^4]\,du\\ &= \frac{1}{9}u^9-\frac{2}{7}u^7+\frac{1}{5}u^5+C\\ &= \frac{1}{9}\sec^9(x)-\frac{2}{7}\sec^7(x)+\frac{1}{5}\sec^5(x)+C.\end{aligned}$$

(b) We rewrite the integral as

$$\begin{aligned}\int \frac{\cos^2(x)}{\sin^6(x)}\,dx &= \int \frac{1}{\sin^4(x)}\cdot\frac{\cos^2(x)}{\sin^2(x)}\,dx\\ &= \int \csc^4(x)\cot^2(x)\,dx\\ &= \int \csc^2(x)\cot^2(x)\csc^2(x)\,dx\\ &= \int [1+\cot^2(x)]\cot^2(x)\csc^2(x)\,dx.\end{aligned}$$

Now let $u = \cot(x)$ so $-du = \csc^2(x)\,dx$. The integral becomes

$$\begin{aligned}\int \frac{\cos^2(x)}{\sin^6(x)}\,dx &= -\int [1+u^2]u^2\,du \\ &= -\int [u^2+u^4]\,du \\ &= -\left[\frac{1}{3}u^3 + \frac{1}{5}u^5\right] + C \\ &= -\frac{1}{3}\cot^3(x) - \frac{1}{5}\cot^5(x) + C.\end{aligned}$$