MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 2.3 | **Math 1001 Worksheet** | WINTER 2024

SOLUTIONS

1. (a) $F'(x) = (x^2+3)^{\cos(x)}$

   (b) Let $u = \tan(x^2)$ so $F(u) = \int_0^u t\,dt$ and thus
   $$F'(x) = F'(u)u' = u[2x\sec^2(x^2)] = 2x\tan(x^2)\sec^2(x^2).$$

   (c) Let $u = e^x$ so
   $$F(u) = \int_u^{100} e^t\,dt = -\int_{100}^u e^t\,dt.$$
   Therefore
   $$F'(x) = F'(u)u' = -e^u[e^x] = -e^x e^{e^x}.$$

   (d) First observe that
   $$\int_{x^3}^x \sqrt{t}\,dt = \int_{x^3}^0 \sqrt{t}\,dt + \int_0^x \sqrt{t}\,dt = -\int_0^{x^3} \sqrt{t}\,dt + \int_0^x \sqrt{t}\,dt.$$
   We can differentiate the second integral using the Fundamental Theorem directly:
   $$\frac{d}{dx}\left[\int_0^x \sqrt{t}\,dt\right] = \sqrt{x}.$$
   For the first integral, let $u = x^3$ so then
   $$\frac{d}{dx}\left[-\int_0^{x^3} \sqrt{t}\,dt\right] = \frac{d}{du}\left[-\int_0^u \sqrt{t}\,dt\right]\frac{du}{dx} = -\sqrt{u}[3x^2] = -3x^2\sqrt{x^3} = -3x^{\frac{7}{2}}.$$
   Now we see that
   $$F'(x) = -3x^{\frac{7}{2}} + \sqrt{x}.$$

2. (a) $\int_0^2 \frac{x^3}{4}\,dx = \frac{1}{4}\left[\frac{x^4}{4}\right]_0^2 = 1$

   (b) $\int_2^3 (2-7x)\,dx = \int_2^3 2\,dx - 7\int_2^3 x\,dx = \Big[2x\Big]_2^3 - 7\left[\frac{x^2}{2}\right]_2^3 = -\frac{31}{2}$

3. (a) This is an elementary integral:

$$\int_1^e (3x^{-3} + 5x^{-1} - 6x^2)\, dx = \left[-\frac{3}{2}x^{-2} + 5\ln|x| - 2x^3\right]_1^e$$
$$= \left[-\frac{3}{2}e^{-2} + 5 - 2e^3\right] - \left[-\frac{3}{2} + 0 - 2\right]$$
$$= \frac{17}{2} - \frac{3}{2e^2} - 2e^3.$$

(b) We expand the product and integrate:

$$\int_0^1 (x+1)(2x-3)\, dx = \int_0^1 (2x^2 - x - 3)\, dx$$
$$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^2 - 3x\right]_0^1$$
$$= -\frac{17}{6}.$$

(c) Since this is the integral of a simple function with linear composition, we have

$$\int_{\frac{\pi}{8}}^{\pi} \cos(2x)\, dx = \left[\frac{1}{2}\sin(2x)\right]_{\frac{\pi}{8}}^{\pi}$$
$$= \frac{1}{2}\left[\sin(2\pi) - \sin\left(\frac{\pi}{4}\right)\right]$$
$$= \frac{1}{2}\left[0 - \frac{\sqrt{2}}{2}\right]$$
$$= -\frac{\sqrt{2}}{4}.$$

(d) This is also the integral of a simple function with linear composition:

$$\int_2^0 (4t+1)^{-\frac{5}{2}}\, dt = \left[\frac{1}{4} \cdot \frac{(4t+1)^{-\frac{3}{2}}}{-\frac{2}{3}}\right]_2^0$$
$$= -\frac{1}{6}\left[1^{-\frac{3}{2}} - 9^{-\frac{3}{2}}\right]$$
$$= -\frac{1}{6}\left[1 - \frac{1}{27}\right]$$
$$= -\frac{13}{81}.$$

(e) The integrand is an improper rational function so we could proceed via long division. However, the similarity between the numerator and the denominator lets us take a short-cut:

$$\begin{aligned}
\int_{-2}^{0} \frac{3x+8}{3x+7}\,dx &= \int_{-2}^{0} \frac{(3x+7)+1}{3x+7}\,dx \\
&= \int_{-2}^{0} \left(1 + \frac{1}{3x+7}\right) dx \\
&= \left[x + \frac{1}{3}\ln|3x+7|\right]_{-2}^{0} \\
&= \left[0 + \frac{1}{3}\ln(7)\right] - \left[-2 + \frac{1}{3}\ln(1)\right] \\
&= 2 + \frac{\ln(7)}{3}.
\end{aligned}$$

(f) We simplify the integrand using long division:

$$\begin{array}{rl}
 & -x+3 \\
x^2+1 & \overline{\big)\, -x^3+3x^2 \qquad\quad +3} \\
 & \;\; -x^3 \qquad\quad -x \\
 & \overline{\qquad\quad 3x^2+x+3} \\
 & \qquad\quad 3x^2 \qquad +3 \\
 & \overline{\qquad\qquad\qquad x\quad}
\end{array}$$

so

$$\frac{3+3x^2-x^3}{x^2+1} = -x+3+\frac{x}{x^2+1}.$$

The integral becomes

$$\int \frac{3+3x^2-x^3}{x^2+1}\,dx = \int_{-2}^{2} (-x+3)\,dx + \int_{-2}^{2} \frac{x}{x^2+1}\,dx.$$

We can handle the first integral using elementary techniques, but we must use u-substitution to evaluate the second integral. Let $u = x^2+1$ so $\frac{1}{2}\,du = x\,dx$. When $x=-2$, $u=5$. When $x=2$, $u=5$ as well. Hence the integral now becomes

$$\begin{aligned}
\frac{3+3x^2-x^3}{x^2+1}\,dx &= \int_{-2}^{2} (-x+3)\,dx + \frac{1}{2}\int_{5}^{5} \frac{du}{u} \\
&= \left[-\frac{1}{2}x^2 + 3x\right]_{-2}^{2} + 0 \\
&= 12.
\end{aligned}$$

(g) Let $u = \ln(x+2)$ so $du = \dfrac{dx}{x+2}$. When $x = -1$, $u = \ln(1) = 0$. When $x = e^3 - 2$, $u = \ln(e^3) = 3$. The integral becomes

$$\begin{aligned}\int_{-1}^{e^3-2} \frac{\ln(x+2)}{x+2}\,dx &= \int_0^3 u\,du \\ &= \left[\frac{1}{2}u^2\right]_0^3 \\ &= \frac{9}{2}.\end{aligned}$$

(h) Let $u = \cos(\theta)$ so $-du = \sin(\theta)\,d\theta$. When $x = 0$, $u = \cos(0) = 1$. When $x = \pi$, $u = \cos(\pi) = -1$. The integral becomes

$$\begin{aligned}\int_0^\pi \cos(\cos(\theta))\sin(\theta)\,d\theta &= -\int_1^{-1} \cos(u)\,du \\ &= -\Big[\sin(u)\Big]_1^{-1} \\ &= \sin(1) - \sin(-1) \\ &= 2\sin(1)\end{aligned}$$

where we have simplified our answer using the fact that $\sin(-x) = -\sin(x)$ for all x.

(i) Let $u = 2 + \dfrac{1}{x}$ so $-du = \dfrac{dx}{x^2}$. When $x = 1$, $u = 3$. When $x = 4$, $u = \frac{9}{4}$. The integral becomes

$$\begin{aligned}\int_{\frac{1}{2}}^4 \frac{1}{x^2}\sqrt{2+\frac{1}{x}}\,dx &= -\int_4^{\frac{9}{4}} \sqrt{u}\,du \\ &= -\left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^{\frac{9}{4}} \\ &= \frac{37}{12}.\end{aligned}$$

(j) Let $u = -\cot\left(\dfrac{x}{3}\right)$ so $3\,du = \csc^2\left(\dfrac{x}{3}\right)\,dx$. When $x = \frac{\pi}{2}$, $u = -\cot\left(\frac{\pi}{6}\right) = -\sqrt{3}$. When $x = 3\pi$, $u = -\cot\left(\frac{\pi}{2}\right) = 0$. The integral becomes

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \csc^2\left(\frac{x}{3}\right)\left[1 - e^{-\cot\left(\frac{x}{3}\right)}\right]\,dx &= 3\int_{-\sqrt{3}}^0 (1 - e^u)\,du \\ &= 3\Big[u - e^u\Big]_{-\sqrt{3}}^0 \\ &= 3\sqrt{3} - 3 + 3e^{-\sqrt{3}}.\end{aligned}$$

(k) Let $u = k^3 - x^3$ so $-\frac{1}{3}\,du = x^2\,dx$. When $x = 0$, $u = k^3$. When $x = k$, $u = 0$. The integral becomes

$$\begin{aligned}\int_0^k x^2(k^3 - x^3)^{\frac{4}{3}}\,dx &= -\frac{1}{3}\int_{k^3}^0 u^{\frac{4}{3}}\,du \\ &= -\frac{1}{3}\left[\frac{3}{7}u^{\frac{7}{3}}\right]_{k^3}^0 \\ &= \frac{1}{7}k^7.\end{aligned}$$

(l) Note that we can write

$$\int_{\sqrt{3}}^{\sqrt{2}} \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx = \int_{\sqrt{3}}^{\sqrt{2}} \frac{1}{\sqrt{1^2 - \left(\frac{x}{2}\right)^2}}\,dx,$$

so we let $u = \frac{x}{2}$ and therefore $2\,du = dx$. When $x = \sqrt{3}$, $u = \frac{\sqrt{3}}{2}$. When $x = \sqrt{2}$, $u = \frac{\sqrt{2}}{2}$. The integral becomes

$$\begin{aligned}\int_{\sqrt{3}}^{\sqrt{2}} \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx &= 2\int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1^2-u^2}}\,du \\ &= 2\Big[\arcsin(u)\Big]_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}} \\ &= -\frac{\pi}{6}.\end{aligned}$$

(m) Let $u = \sqrt{t}$ so $2\,du = t^{-\frac{1}{2}}\,dt$ and $t = u^2$. When $t = 0$, $u = 0$. When $t = 4$, $u = 2$. The integral becomes

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{t}(t+4)}\,dt &= 2\int_0^2 \frac{1}{u^2+2^2}\,du \\ &= 2\left[\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right]_0^2 \\ &= \frac{\pi}{4}.\end{aligned}$$

(n) Note first that

$$\int_{\sqrt{e}}^{e} \frac{1}{x\ln(x)\sqrt{16(\ln(x))^2-4}}\,dx = \int_{\sqrt{e}}^{e} \frac{1}{x\ln(x)\sqrt{(4\ln(x))^2-2^2}}\,dx.$$

Then let $u = 4\ln(x)$ so $\dfrac{1}{4}\,du = \dfrac{dx}{x}$ and $\frac{1}{4}u = \ln(x)$. When $x = \sqrt{e}$, $u = 4\ln(\sqrt{e}) = 4\ln(e^{\frac{1}{2}}) = 2$. When $x = e$, $u = 4\ln(e) = 4$. The integral becomes

$$\begin{aligned}\int_{\sqrt{e}}^{e} \frac{1}{x\ln(x)\sqrt{16(\ln(x))^2 - 4}}\,dx &= \frac{1}{4}\int_2^4 \frac{1}{\left(\frac{u}{4}\right)\sqrt{u^2 - 2^2}}\,du \\ &= \int_2^4 \frac{1}{u\sqrt{u^2-2^2}}\,du \\ &= \left[\frac{1}{2}\operatorname{arcsec}\left(\frac{u}{2}\right)\right]_2^4 \\ &= \frac{\pi}{6}.\end{aligned}$$

(o) We use integration by parts. Let $w = \ln(x)$ so $dw = x^{-1}\,dx$, and let $dv = x^{-4}\,dx$ so $v = -\frac{1}{3}x^{-3}$. Then

$$\begin{aligned}\int_1^3 \frac{\ln(x)}{x^4}\,dx &= \left[-\frac{\ln(x)}{3x^3}\right]_1^3 + \frac{1}{3}\int_1^3 x^{-4}\,dx \\ &= \left[-\frac{\ln(x)}{3x^3} - \frac{1}{9x^3}\right]_1^3 \\ &= \frac{26}{243} - \frac{\ln(3)}{81}.\end{aligned}$$

(p) Let $u = \sin(x)$ so $du = \cos(x)\,dx$. When $x = \arcsin\left(\frac{3}{5}\right)$, $u = \frac{3}{5}$. When $x = \frac{\pi}{2}$, $u = 1$. Thus the integral becomes

$$\int_{\arcsin\left(\frac{3}{5}\right)}^{\frac{\pi}{2}} \cos(x)\ln(\sin(x))\,dx = \int_{\frac{3}{5}}^{1} \ln(u)\,du.$$

Now we use integration by parts. Let $w = \ln(u)$ so $dw = \frac{1}{u}\,du$. Let $dv = du$ so $v = u$. Then

$$\begin{aligned}\int_{\arcsin\left(\frac{3}{5}\right)}^{\frac{\pi}{2}} \cos(x)\ln(\sin(x))\,dx &= \Big[u\ln(u)\Big]_{\frac{3}{5}}^{1} - \int_{\frac{3}{5}}^{1} u\cdot\frac{1}{u}\,du \\ &= \Big[u\ln(u)\Big]_{\frac{3}{5}}^{1} - \int_{\frac{3}{5}}^{1} du \\ &= \Big[u\ln(u) - u\Big]_{\frac{3}{5}}^{1} \\ &= -\frac{2}{5} - \frac{3}{5}\ln\left(\frac{3}{5}\right).\end{aligned}$$

(q) First we make the substitution $u = x^2$ so $\frac{1}{2}\,du = x\,dx$. When $x = 0$, $u = 0$. When $x = 1$, $u = 1$. The integral becomes

$$\int_0^1 x \arcsin(x^2)\,dx = \frac{1}{2}\int_0^1 \arcsin(u)\,du.$$

Now we use integration by parts. Let $w = \arcsin(u)$ so $dw = \dfrac{1}{\sqrt{1-u^2}}\,du$, and let $dv = du$ so $v = u$. Then

$$\int_0^1 x \arcsin(x^2)\,dx = \frac{1}{2}\Big[u \arcsin(u)\Big]_0^1 - \frac{1}{2}\int_0^1 \frac{u}{\sqrt{1-u^2}}\,du.$$

Now we have to make another substitution: let $z = 1 - u^2$ so $-\frac{1}{2}\,dz = u\,du$. When $u = 0$, $z = 1$. When $u = 1$, $z = 0$. So then

$$\begin{aligned}\int_0^1 x \arcsin(x^2)\,dx &= \frac{1}{2}\Big[u \arcsin(u)\Big]_0^1 + \frac{1}{4}\int_1^0 z^{-\frac{1}{2}}\,dz \\ &= \left[\frac{1}{2}u \arcsin(u)\right]_0^1 + \frac{1}{2}\Big[\sqrt{z}\Big]_1^0 \\ &= \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

(r) First observe that we can write

$$|2x+8| = \begin{cases} 2x+8 & \text{for } x \geq -4 \\ -(2x+8) & \text{for } x < -4. \end{cases}$$

Using the Additive Interval Property, then, we can write

$$\begin{aligned}\int_{-5}^{-1} |2x+8|\,dx &= \int_{-5}^{-4} |2x+8|\,dx + \int_{-4}^{-1} |2x+8|\,dx \\ &= -\int_{-5}^{-4} (2x+8)\,dx + \int_{-4}^{-1} (2x+8)\,dx \\ &= -\Big[x^2+8x\Big]_{-5}^{-4} + \Big[x^2+8x\Big]_{-4}^{-1} \\ &= 10.\end{aligned}$$

(s) Again, we begin by rewriting the absolute value as a piecewise function; because the function inside the absolute value is more complicated than those we have previously considered, we need to do a little more work to see how to write it in this form. We are interested in the intervals where $x^2 - 4x + 3$ is positive or negative, so we set

$$x^2 - 4x + 3 = (x-3)(x-1) = 0,$$

giving $x = 1$ or $x = 3$. This means that there are three intervals of interest: $x < 1$, $1 < x < 3$ and $x > 3$. The function $x^2 - 4x + 3$ is either positive for all x, or negative for all x, on each of these intervals. We just need to test a value of x in each interval to see which is which. For $x < 1$, try $x = 0$. Then

$$x^2 - 4x + 3 = 0^2 - 4(0) + 3 = 3,$$

so $x^2 - 4x + 3 > 0$ for $x < 1$ (which means that the absolute value has no effect). For $1 < x < 3$, try $x = 2$. Then

$$x^2 - 4x + 3 = 2^2 - 4(2) + 3 = -1,$$

so $x^2 - 4x + 3 < 0$ for $1 < x < 3$ and therefore

$$|x^2 - 4x + 3| = -(x^2 - 4x + 3)$$

on this interval. Finally, for $x > 3$, we try $x = 4$. Then

$$x^2 - 4x + 3 = 4^2 - 4(4) + 3 = 3,$$

so $x^2 - 4x + 3 > 0$ for $x > 3$ and, again, the absolute value does nothing. Hence the piecewise definition of the integrand is

$$|x^2 - 4x + 3| = \begin{cases} x^2 - 4x + 3 & \text{for } x \leq 1 \text{ and } x \geq 3 \\ -(x^2 - 4x + 3) & \text{for } 1 < x < 3. \end{cases}$$

Finally, using the Additive Interval Property, we obtain

$$\begin{aligned} &\int_{-1}^{4} |x^2 - 4x + 3| \, dx \\ &= \int_{-1}^{1} |x^2 - 4x + 3| \, dx + \int_{1}^{3} |x^2 - 4x + 3| \, dx + \int_{3}^{4} |x^2 - 4x + 3| \, dx \\ &= \int_{-1}^{1} (x^2 - 4x + 3) \, dx - \int_{1}^{3} (x^2 - 4x + 3) \, dx + \int_{3}^{4} (x^2 - 4x + 3) \, dx \\ &= \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_{-1}^{1} - \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_{1}^{3} + \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_{3}^{4} \\ &= \frac{28}{3}. \end{aligned}$$

4. $A = \displaystyle\int_{\frac{11}{4}}^{\frac{35}{4}} \frac{2}{\sqrt{2x - \frac{3}{2}}} \, dx = 2\left[\frac{1}{2} \cdot \frac{\sqrt{2x - \frac{3}{2}}}{\frac{1}{2}}\right]_{\frac{11}{4}}^{\frac{35}{4}} = 2[4 - 2] = 4$