Section 1.4

Integration by parts: $\int w\,dv = vw - \int v\,dw$

When choosing dv, we need an expression that we can integrate, ideally an elementary integral. For w, we would like an expression that becomes simpler when it's differentiated, such as x^n (for n a natural number), a logarithmic function or an inverse trigonometric function.

eg $\int xe^x\,dx$

We use integration by parts with

$$w = x \qquad dw = 1 \cdot dx = dx$$

$$dv = e^x\,dx \qquad v = e^x$$

so
$$\int xe^x\,dx = xe^x - \int e^x\,dx$$
$$\boxed{= xe^x - e^x + C}$$

Note that v need only be an antiderivative of dv because if we include a constant of integration, we would get

$$\begin{aligned}
\int w\,dv &= (v+C)w - \int (v+C)\,dw \\
&= vw + Cw - \int v\,dw - C\int dw \\
&= vw + Cw - \int v\,dw - Cw \\
&= vw - \int v\,dw
\end{aligned}$$

In some cases, integration by parts must be applied several times in order to evaluate a given integral.

eg $\int x^2 e^{6x}\,dx$

We use integration by parts with

$$w = x^2 \qquad dw = 2x\,dx$$
$$dv = e^{6x}\,dx \qquad v = \frac{1}{6}e^{6x}$$

so we have

$$\int x^2 e^{6x}\,dx = \frac{1}{6}x^2e^{6x} - \int \left(\frac{1}{6}e^{6x}\right)\cdot 2x\,dx$$
$$= \frac{1}{6}x^2e^{6x} - \frac{1}{3}\int xe^{6x}\,dx$$

Now we use integration by parts again with

$$w = x \qquad dw = 1\cdot dx = dx$$
$$dv = e^{6x}\,dx \qquad v = \frac{1}{6}e^{6x}$$

which gives

$$\int x^2e^{6x}\,dx = \frac{1}{6}x^2e^{6x} - \frac{1}{3}\left[\frac{1}{6}xe^{6x} - \int \frac{1}{6}e^{6x}\,dx\right]$$
$$= \frac{1}{6}x^2e^{6x} - \frac{1}{18}xe^{6x} + \frac{1}{18}\int e^{6x}\,dx$$
$$= \frac{1}{6}x^2e^{6x} - \frac{1}{18}xe^{6x} + \frac{1}{18}\left[\frac{1}{6}e^{6x}\right] + C$$
$$\boxed{= \frac{1}{6}x^2e^{6x} - \frac{1}{18}xe^{6x} + \frac{1}{108}e^{6x} + C}$$

eg $\int e^x \cos(x)\,dx$

We use integration by parts with

$w = \cos(x)$ $\quad\quad dw = -\sin(x)\,dx$

$dv = e^x\,dx$ $\quad\quad v = e^x$

so $\int e^x \cos(x)\,dx = e^x\cos(x) - \int e^x \cdot [-\sin(x)\,dx]$

$= e^x\cos(x) + \int e^x \sin(x)\,dx$

We use integration by parts again, with

$w = \sin(x)$ $\quad\quad dw = \cos(x)\,dx$

$dv = e^x\,dx$ $\quad\quad v = e^x$

so $\int e^x \cos(x)\,dx = e^x\cos(x) + \left[e^x\sin(x) - \int e^x\cos(x)\,dx\right]$

$= e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)\,dx$

$2\int e^x\cos(x)\,dx = e^x\cos(x) + e^x\sin(x)$

$\int e^x\cos(x)\,dx \boxed{= \frac{1}{2}e^x\cos(x) + \frac{1}{2}e^x\sin(x) + C}$

Sometimes we will apply integration by parts to $\int f(x)\,dx$ by treating as $\int f(x)\cdot 1\,dx$ and setting

$w = f(x)$ and $dv = 1\cdot dx = dx$

eg $\int \ln(x)\,dx$

Let $w = \ln(x)$ $\qquad dw = \frac{1}{x}\,dx$

$dv = dx$ $\qquad v = x$

So, using integration by parts,

$$\int \ln(x)\,dx = x\ln(x) - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln(x) - \int dx$$

$$\boxed{= x\ln(x) - x + C}$$

We often need to combine u-substitution and integration by parts.

eg $\int \arctan(x)\,dx$

Let $w = \arctan(x)$ $\qquad dw = \frac{1}{x^2+1}\,dx$

$dv = dx$ $\qquad v = x$

So we have, by integration by parts,

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{x^2+1}\,dx$$

To evaluate the remaining integral, we use u-substitution with

$u = x^2+1$ $\qquad du = 2x\,dx$

$\qquad \frac{1}{2}du = x\,dx$

Now we have

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{1}{u} \cdot \frac{1}{2}\,du$$

$$= x\arctan(x) - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= x\arctan(x) - \frac{1}{2}\ln|u| + C$$

$$\boxed{= x\arctan(x) - \frac{1}{2}\ln(x^2+1) + C}$$