Section 1.3

eg (cont.)

$$\int \frac{1}{(x-1)\sqrt{x^2-2x-4}}\,dx$$

$$= \int \frac{1}{(x-1)\sqrt{(x-1)^2-5}}\,dx$$

Let $u = x-1$

$du = 1 \cdot dx = dx$

The integral becomes

$$\int \frac{1}{(x-1)\sqrt{x^2-2x-4}}\,dx = \int \frac{1}{u\sqrt{u^2-5}}\,du$$

$$= \frac{1}{\sqrt{5}} \operatorname{arcsec}\left(\frac{u}{\sqrt{5}}\right) + C$$

$$\boxed{= \frac{\sqrt{5}}{5} \operatorname{arcsec}\left(\frac{\sqrt{5}}{5}(x-1)\right) + C}$$

To complete the square for an expression of the form

$$ax^2+bx+c$$

where $a \neq 1$, we first factor a outside the entire trinomial. Then we complete the square for the remaining quadratic expression, and multiply a back in.

eg $\displaystyle\int \frac{1}{4x^2+12x+10}\,dx$

We complete the square:

$$4x^2+12x+10 = 4\left[x^2+3x+\frac{5}{2}\right]$$

$$= 4\left[(x^2+3x)+\frac{5}{2}\right]$$

$$= 4\left[\left(x^2+3x+\frac{9}{4}\right)+\frac{5}{2}-\frac{9}{4}\right]$$

$$= 4\left[\left(x+\frac{3}{2}\right)^2+\frac{1}{4}\right]$$

$$= 4\left(x+\frac{3}{2}\right)^2+1$$

$$= 2^2\left(x+\frac{3}{2}\right)^2+1$$

$$= (2x+3)^2+1$$

The integral can be rewritten as

$$\int \frac{1}{4x^2+12x+10}\,dx = \int \frac{1}{(2x+3)^2+1}\,dx$$

Let $u = 2x+3$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \int \frac{1}{u^2+1}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\cdot\frac{1}{1}\arctan\left(\frac{u}{1}\right)+C$$

$$\boxed{= \frac{1}{2}\arctan(2x+3)+C}$$

eg $\displaystyle\int \frac{1}{\sqrt{20+8x-x^2}}\,dx$

We write
$$\begin{aligned}20+8x-x^2 &= -\left[x^2-8x-20\right]\\ &= -\left[(x^2-8x)-20\right]\\ &= -\left[(x^2-8x+16)-20-16\right]\\ &= -\left[(x-4)^2-36\right]\\ &= 36-(x-4)^2\end{aligned}$$

Now the integral becomes

$$\int \frac{1}{\sqrt{20+8x-x^2}}\,dx = \int \frac{1}{\sqrt{36-(x-4)^2}}\,dx \qquad \begin{aligned}\text{Let } u &= x-4\\ du &= dx\end{aligned}$$

$$= \int \frac{1}{\sqrt{36-u^2}}\,du$$

$$= \arcsin\left(\frac{u}{6}\right) + C$$

$$\boxed{= \arcsin\left(\frac{x-4}{6}\right) + C}$$

Very rarely, the process of integration will naturally lead to an arccosine, arccotangent or arccosecant function, usually when one or more is present in the integrand.

eg $\displaystyle\int \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx$

Let $u = \arccos(x)$ so $du = \dfrac{-1}{\sqrt{1-x^2}}\,dx$

$$-du = \frac{1}{\sqrt{1-x^2}}\,dx$$

The integral becomes

$$\int \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx = \int u \cdot (-du)$$

$$= -\int u\,du$$

$$= -\left[\frac{u^2}{2}\right] + C$$

$$\boxed{= -\frac{1}{2}\arccos^2(x) + C}$$

Section 1.4: Integration by Parts

The Product Rule states that

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

This means that

$$\int [f'(x)g(x) + f(x)g'(x)]\,dx = f(x)g(x) + C$$

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x) + C$$

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

Let $v = f(x)$ and $w = g(x)$. Then $dw = f'(x)\,dx$ and $dw = g'(x)\,dx$. Then we have

$$\int g(x) \cdot f'(x)\,dx = f(x)g(x) - \int f(x) \cdot g'(x)\,dx$$

$$\int w\,dv = vw - \int v\,dw$$

which is the <u>integration by parts</u> formula.

eg $\int x \ln(x)\,dx$

One approach to integration by parts is to let

$w = x \qquad dw = dx$

$dv = \ln(x)\,dx \qquad v = \int \ln(x)\,dx$

which we cannot evaluate

Instead, we will try

$w = \ln(x) \qquad dw = \frac{1}{x}\,dx$

$dv = x\,dx \qquad v = \int x\,dx = \frac{1}{2}x^2$

By the integration by parts formula, we can write

$$\int x \ln(x)\,dx = \frac{1}{2}x^2 \ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\left[\frac{x^2}{2}\right] + C$$

$$\boxed{= \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C}$$