

Section 1.3

We now have $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$.

More generally, we have the following:

Theorem: $\int \frac{1}{\sqrt{k^2-x^2}} dx = \arcsin\left(\frac{x}{k}\right) + C$
for any constant $k > 0$

Proof: We can rewrite the integral as

$$\begin{aligned}\int \frac{1}{\sqrt{k^2-x^2}} dx &= \int \frac{1}{\sqrt{k^2\left(1-\frac{x^2}{k^2}\right)}} dx \\ &= \int \frac{1}{\sqrt{k^2} \cdot \sqrt{1-\frac{x^2}{k^2}}} dx \\ &= \int \frac{1}{k\sqrt{1-\frac{x^2}{k^2}}} dx \\ &= \int \frac{1}{k\sqrt{1-\left(\frac{x}{k}\right)^2}} dx\end{aligned}$$

Let $u = \frac{x}{k}$ so $du = \frac{1}{k} dx$ and the integral

becomes

$$\begin{aligned}\int \frac{1}{\sqrt{k^2-x^2}} dx &= \int \frac{1}{\sqrt{1-u^2}} du \\ &= \arcsin(u) + C \\ &= \arcsin\left(\frac{x}{k}\right) + C.\end{aligned}$$

$$\text{eg } \int \frac{1}{\sqrt{25-x^2}} dx = \int \frac{1}{\sqrt{5^2-x^2}} dx$$

This is an arcsine integral with $k=5$. Thus

$$\int \frac{1}{\sqrt{25-x^2}} dx = \boxed{\arcsin\left(\frac{x}{5}\right) + C}$$

Theorem: $\int \frac{1}{x^2+k^2} dx = \frac{1}{k} \arctan\left(\frac{x}{k}\right) + C$

$$\text{eg } \int \frac{e^x}{e^{2x}+1} dx = \int \frac{e^x}{(e^x)^2+1} dx$$

Let $u=e^x$ then $du=e^x dx$ so the integral becomes

$$\int \frac{e^x}{e^{2x}+1} dx = \int \frac{1}{u^2+1} du$$

which is an arctangent integral with $k=1$. Hence

$$\int \frac{e^x}{e^{2x}+1} dx = \frac{1}{1} \arctan\left(\frac{u}{1}\right) + C$$
$$= \boxed{\arctan(e^x) + C}$$

Theorem: $\int \frac{1}{x\sqrt{x^2-k^2}} dx = \frac{1}{k} \operatorname{arcsec}\left(\frac{x}{k}\right) + C$

$$\text{eg } \int \frac{1}{x\sqrt{16x^2-9}} dx$$

$$= \int \frac{1}{x\sqrt{16\left(x^2-\frac{9}{16}\right)}} dx$$

$$= \int \frac{1}{x \cdot 4\sqrt{x^2-\frac{9}{16}}} dx$$

$$= \frac{1}{4} \int \frac{1}{x\sqrt{x^2-\left(\frac{3}{4}\right)^2}} dx$$

This is an arcsecant integral
with $k = \frac{3}{4}$

$$= \frac{1}{4} \cdot \frac{1}{\frac{3}{4}} \operatorname{arcsec}\left(\frac{x}{\frac{3}{4}}\right) + C$$

$$\boxed{= \frac{1}{3} \operatorname{arcsec}\left(\frac{4x}{3}\right) + C}$$

Alternatively, we can rewrite the integral as

$$\int \frac{1}{x\sqrt{(4x)^2-9}} dx$$

$$\text{Let } u = 4x$$

$$du = 4dx \rightarrow \frac{1}{4} du = dx$$

$$x = \frac{1}{4}u$$

$$= \int \frac{1}{\frac{1}{4}u\sqrt{u^2-9}} \cdot \frac{1}{4} du$$

$$= \int \frac{1}{u\sqrt{u^2-9}} du$$

This is an arcsecant integral
with $k = 3$

$$= \frac{1}{3} \operatorname{arcsec}\left(\frac{u}{3}\right) + C$$

$$\boxed{= \frac{1}{3} \operatorname{arcsec}\left(\frac{4x}{3}\right) + C}$$

Integrals of rational functions often lead to arctangent functions.

$$\text{eg } \int \frac{x+3}{x^2+9} dx = \int \frac{x}{x^2+9} dx + \int \frac{3}{x^2+9} dx$$

For the first integral, we let $u = x^2 + 9$
 $du = 2x dx$
 $\frac{1}{2} du = x dx$

Thus

$$\begin{aligned} \int \frac{x}{x^2+9} dx &= \int \frac{1}{u} \cdot \frac{1}{2} du \\ &= \frac{1}{2} \int \frac{1}{u} du \\ &= \frac{1}{2} \ln|u| + C \\ &= \frac{1}{2} \ln|x^2+9| + C \\ &= \frac{1}{2} \ln(x^2+9) + C \end{aligned}$$

For the second integral, this is an arctangent form with $k=3$ so

$$\begin{aligned} \int \frac{3}{x^2+9} dx &= 3 \int \frac{1}{x^2+9} dx \\ &= 3 \cdot \frac{1}{3} \arctan\left(\frac{x}{3}\right) + C \\ &= \arctan\left(\frac{x}{3}\right) + C \end{aligned}$$

$$\text{Finally, } \int \frac{x+3}{x^2+9} dx = \boxed{\frac{1}{2} \ln(x^2+9) + \arctan\left(\frac{x}{3}\right) + C}$$

In order to apply an inverse trigonometric integral to a case where the quadratic expression in the denominator includes a linear term, we must first complete the square.

Suppose we have $x^2 + bx + c$ where b and c are constants. Then we first rewrite this as

$$\begin{aligned}(x^2 + bx) + c &= \left(x^2 + bx + \left(\frac{b}{2}\right)^2\right) + \left(c - \left(\frac{b}{2}\right)^2\right) \\ &= \left(x + \frac{b}{2}\right)^2 + \left(c - \left(\frac{b}{2}\right)^2\right)\end{aligned}$$

eg $\int \frac{1}{(x-1)\sqrt{x^2-2x-4}} dx$

We start by completing the square:

$$\begin{aligned}x^2 - 2x - 4 &= (x^2 - 2x) - 4 \\ &= (x^2 - 2x + 1) - 4 - 1 \\ &= (x-1)^2 - 5\end{aligned}$$

The integral becomes

$$\int \frac{1}{(x-1)\sqrt{x^2-2x-4}} dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-5}} dx$$