Section 1.2

eg (cont.) $\displaystyle\int \frac{x^2+1}{x-3}\,dx$

We will use long division to rewrite the improper rational function as the <u>sum</u> of a <u>polynomial</u> and a <u>proper</u> rational function.

(1) (2) (3) (4)

$$\begin{array}{r l}
 & x + 3 \\ \cline{2-2}
x-3 & \big)\, x^2 \quad\quad + 1 \\
 & \underline{x^2 - 3x} \\
 & \quad\; 3x + 1 \\
 & \quad\; \underline{3x - 9} \\
 & \quad\quad\;\; 10
\end{array}$$

1. Write the dividend and divisor in descending order of powers of x
2. What multiplies x to get x^2?
3. Multiply this expression by the divisor
4. Subtract this expression from the dividend to obtain the remainder
5. Repeat (2)-(4) with the remainder as the new dividend until its degree is smaller than that of the divisor

The quotient obtained is the desired polynomial. The proper rational function is the remainder, divided by the original denominator.

Thus,

$$\frac{x^2+1}{x-3} = x+3 + \frac{10}{x-3}$$

$$\int \frac{x^2+1}{x-3}\,dx = \int \left(x+3 + \frac{10}{x-3}\right)dx$$

$$= \frac{1}{2}x^2 + 3x + 10 \cdot \frac{\ln|x-3|}{1} + C$$

$$\boxed{= \frac{1}{2}x^2 + 3x + 10\ln|x-3| + C}$$

eg $\displaystyle\int \frac{6x^2-2x-3}{1-2x^2}\,dx$

By long division, we have

$$\begin{array}{r|l} & \quad -3 \\ \hline -2x^2+1 & 6x^2-2x-3 \\ & 6x^2 \qquad -3 \\ \hline & \quad -2x \end{array}$$

Thus $\displaystyle\int \frac{6x^2-2x-3}{1-2x^2}\,dx = \int \left(-3 - \frac{2x}{1-2x^2}\right)dx$

$$= -3x - 2\int \frac{x}{1-2x^2}\,dx$$

Now let $u = 1-2x^2$

$du = -4x\,dx \longrightarrow -\frac{1}{4}\,du = x\,dx$

We have

$$-3x - 2\int \frac{1}{u}\cdot\left(-\frac{1}{4}\,du\right)$$

$$= -3x + \frac{1}{2}\int \frac{1}{u}\,du$$

$$= -3x + \frac{1}{2}\ln|u| + C$$

$$\boxed{= -3x + \frac{1}{2}\ln|1-2x^2| + C}$$

Theorem: ① $\int \tan(x)\,dx = -\ln|\cos(x)| + C$

② $\int \cot(x)\,dx = \ln|\sin(x)| + C$

Proof: ① Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ so

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx.$$

Let $u = \cos(x)$

$du = -\sin(x)\,dx \rightarrow -du = \sin(x)\,dx$

The integral becomes

$$\int \tan(x)\,dx = \int \frac{1}{u}\cdot(-du) = -\int \frac{1}{u}\,du$$
$$= -\ln|u| + C$$
$$= -\ln|\cos(x)| + C.$$

Note that we can instead write

$$\int \tan(x)\,dx = \ln\left|[\cos(x)]^{-1}\right| + C$$
$$= \ln|\sec(x)| + C$$

eg $$\int \tan\left(\frac{x}{5}\right)dx = \frac{-\ln\left|\cos\left(\frac{x}{5}\right)\right|}{1/5} + C$$
$$\boxed{= -5\ln\left|\cos\left(\frac{x}{5}\right)\right| + C}$$

Theorem: ① $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

② $\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$

$= \ln|\csc(x) - \cot(x)| + C$

Proof: ② We rewrite

$$\csc(x) = \csc(x) \cdot \frac{\csc(x) + \cot(x)}{\csc(x) + \cot(x)}$$

$$= \frac{\csc^2(x) + \csc(x)\cot(x)}{\csc(x) + \cot(x)}$$

so $\int \csc(x)\,dx = \int \frac{\csc^2(x) + \csc(x)\cot(x)}{\csc(x) + \cot(x)}\,dx$

Let $u = \csc(x) + \cot(x)$

$du = [-\csc(x)\cot(x) - \csc^2(x)]\,dx$

$-du = [\csc^2(x) + \csc(x)\cot(x)]\,dx$

The integral becomes

$$\int \csc(x)\,dx = \int \frac{1}{u} \cdot (-du)$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\csc(x) + \cot(x)| + C.$$

Section 1.3: Integration and Inverse Trigonometric Functions

Recall that $[\arcsin(x)]' = \frac{1}{\sqrt{1-x^2}}$

$$[\arccos(x)]' = \frac{-1}{\sqrt{1-x^2}}.$$

So we can write

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

and $$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$$

However, we could instead write

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= -\arcsin(x) + C$$

Thus we generally only use $\arcsin(x)$ in our integration results not $\arccos(x)$. The same is true for $\arctan(x)$ and $\operatorname{arcsec}(x)$.