Section 1.2

eg $\displaystyle\int \frac{5x^2}{\sqrt{1-2x^3}}\,dx$

We let $u = 1-2x^3$

$du = -6x^2\,dx$

$-\frac{1}{6}\,du = x^2\,dx$

The integral becomes

$$\int 5 \cdot \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{6}\,du\right) = -\frac{5}{6}\int u^{-1/2}\,du$$

$$= -\frac{5}{6}\left[\frac{u^{1/2}}{1/2}\right] + C$$

$$\boxed{= -\frac{5}{3}\sqrt{1-2x^3} + C}$$

eg $\displaystyle\int 3x^5\sqrt{1+x^3}\,dx$

Let $u = 1+x^3$

$du = 3x^2\,dx$

We can rewrite the integral as

$$\int 3x^5\sqrt{1+x^3}\,dx = \int x^3\sqrt{1+x^3} \cdot 3x^2\,dx$$

But observe that $x^3 = u-1$ so this becomes

$$\int (u-1)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right)du$$

$$\int 3x^5\sqrt{1+x^3}\,dx = \int u^{3/2}\,du - \int u^{1/2}\,du$$

$$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C$$

$$\boxed{= \frac{2}{5}(1+x^3)^{5/2} - \frac{2}{3}(1+x^3)^{3/2} + C}$$

Sometimes we can apply u-substitution to an integral which does not involve a composite function.

eg $\displaystyle\int \frac{\ln(x)}{x}\,dx = \int \ln(x)\cdot\frac{1}{x}\,dx$

Let $u = \ln(x)$

$du = \frac{1}{x}\,dx$

The integral becomes

$$\int \frac{\ln(x)}{x}\,dx = \int u\,du$$

$$= \frac{u^2}{2} + C \quad \boxed{= \frac{[\ln(x)]^2}{2} + C = \frac{\ln^2(x)}{2} + C}$$

Recall that we have previously shown that if $\int f(x)\,dx = F(x) + C$ then

$$\int f(mx+b)\,dx = \frac{1}{m}F(mx+b) + C$$

if $m \neq 0$. We can prove this using u-substitution.

We let $u = mx + b$

$$du = m\,dx$$

$$\frac{1}{m}\,du = dx$$

The integral becomes

$$\int f(u)\cdot\frac{1}{m}\,du = \frac{1}{m}\int f(u)\,du = \frac{1}{m}F(u) + C$$

$$= \frac{1}{m}F(mx+b) + C.$$

eg $\displaystyle\int \frac{2}{5x+9}\,dx$ Let $u = 5x+9$

$du = 5\,dx \rightarrow \frac{1}{5}\,du = dx$

$$= \int \frac{2}{u}\cdot\frac{1}{5}\,du$$

$$= \frac{2}{5}\int\frac{1}{u}\,du$$

$$= \frac{2}{5}\cdot \ln|u| + C \boxed{= \frac{2}{5}\ln|5x+9| + C}$$

In general when integrating a rational function, if we want to apply u-substitution then we will typically let u be the denominator.

eg $\displaystyle\int \frac{3x^2+x+4}{2x^3+x^2+8x+7}\,dx$

Let $u = 2x^3+x^2+8x+7$

$$du = (6x^2+2x+8)\,dx$$

$$\frac{1}{2}\,du = (3x^2+x+4)\,dx$$

The integral becomes

$$\int \frac{3x^2+x+4}{2x^3+x^2+8x+7}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du$$
$$= \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u| + C$$
$$\boxed{= \frac{1}{2}\ln|2x^3+x^2+8x+7| + C}$$

In general, we can divide rational functions into <u>proper</u> and <u>improper</u> rational functions. They all have the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. In a proper rational function, the degree of $P(x)$ is <u>strictly less</u> than the degree of $Q(x)$. Otherwise, it is improper.

eg $f(x) = \frac{x^2+1}{x-3}$ is an improper rational function

$f(x) = \frac{3x^2+x+4}{2x^3+x^2+8x+7}$ is a proper rational function

$f(x) = \frac{5x+4}{2-3x}$ is an improper rational function

Improper rational functions can be rewritten in terms of proper rational functions using <u>long division</u>.

eg $\int \frac{x^2+1}{x-3}\,dx$

Let $u = x-3$

$du = 1 \cdot dx = dx$

Then $x = u+3$

$x^2+1 = (u+3)^2+1 = u^2+6u+10$

The integral becomes

$$\int \frac{x^2+1}{x-3}\,dx = \int \frac{u^2+6u+10}{u}\,du$$

$$= \int u\,du + 6\int du + 10\int \frac{1}{u}\,du$$

$$= \frac{u^2}{2} + 6 \cdot u + 10 \cdot \ln|u| + C$$

$$\boxed{= \frac{1}{2}(x-3)^2 + 6(x-3) + 10 \ln|x-3| + C}$$