Section 1.2: Integration by Substitution

Consider a line l. Its slope is given by

$$m = \frac{\Delta y}{\Delta x}$$

where Δx represents the change in the x-coordinate and Δy represents the corresponding change in the y-coordinate. Now suppose that l is the tangent line to a curve $y = f(x)$ at a point x. Then $m = f'(x)$ so

$$\frac{\Delta y}{\Delta x} = f'(x) \longrightarrow \Delta y = f'(x)\,\Delta x.$$

$$\Delta y = \frac{dy}{dx}\,\Delta x$$

We typically use Δx and Δy to represent a "large" change in the variable. To indicate an infinitesimal change, we instead represent this as dx or dy, and we call these <u>differentials</u>. Hence

$$dy = \frac{dy}{dx}\,dx$$

This lets us rewrite an integral given with respect to one variable in terms of a different variable, because it shows us how to rewrite the differential.

Recall that the Chain Rule (for derivatives) is given by

$$[f(g(x))]' = f'(g(x))\, g'(x)$$

We therefore have

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

We can let $u = g(x)$ so then $\frac{du}{dx} = g'(x)$.

The integral can be rewritten as

$$\int f'(u) \cdot \frac{du}{dx} \cdot dx = f(u) + C$$

But $du = \frac{du}{dx} \cdot dx$ so this becomes

$$\int f'(u)\, du = f(u) + C$$

This is called <u>integration by substitution</u> or <u>u-substitution</u>.

eg $\int 2x\sqrt{x^2+1}\, dx$

We let $u = x^2 + 1$

$$\frac{du}{dx} = 2x \rightarrow du = 2x\, dx$$

Thus we can write

$$\int 2x\sqrt{x^2+1}\, dx = \int \sqrt{x^2+1} \cdot 2x\, dx = \int \sqrt{u}\, du$$

$$\int 2x\sqrt{x^2+1}\,dx = \int u^{1/2}\,du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$

$$\boxed{= \frac{2}{3}(x^2+1)^{3/2} + C}$$

General steps for u-substitution:

1. Identify an appropriate expression for u (often the inside function in a composite function)
2. Rewrite all expressions in the integral, including the differential, in terms of u.
3. Integrate the resulting expression, which will hopefully now be an elementary integral.
4. Substitute back in for u to write the answer in terms of the original variable.

eg $\int \cos(x)\, e^{\sin(x)}\,dx$

Let $u = \sin(x)$

$du = \cos(x)\,dx$

The integral becomes

$$\int \cos(x) e^{\sin(x)} dx = \int e^u du$$

$$= e^u + C$$

$$\boxed{= e^{\sin(x)} + C}$$

eg $\int 4\tan^3(\theta) \sec^2(\theta)\, d\theta$

Let $u = \tan(\theta)$

$du = \sec^2(\theta)\, d\theta$

The integral becomes

$$\int 4\tan^3(\theta)\sec^2(\theta)\, d\theta = \int 4u^3\, du$$

$$= 4\int u^3\, du$$

$$= 4\left[\frac{u^4}{4}\right] + C$$

$$\boxed{= \tan^4(\theta) + C}$$

eg $\int x^3 \cosh(x^4+2)\, dx$

Let $u = x^4 + 2$

$du = 4x^3\, dx$

But note that $\int x^3 \cosh(x^4+2)\, dx = \int \frac{1}{4}\cdot 4x^3 \cosh(x^4+2)\, dx$

More simply, we can write

$$\frac{1}{4}\,du = x^3\,dx$$

The integral becomes

$$\int x^3 \cosh(x^4+2)\,dx = \int \cosh(u)\cdot\frac{1}{4}\,du$$

$$= \frac{1}{4}\int \cosh(u)\,du$$

$$= \frac{1}{4}\left[\sinh(u)\right] + C$$

$$\boxed{= \frac{1}{4}\sinh(x^4+2) + C}$$