Section 4.5

Mean value of a probability density function: $\mu = \int_a^b x f(x)\,dx$

eg Determine the mean total viewing time of two videos, using the probability density function previously established.

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^8 x \cdot \frac{3}{256} x(8-x)\,dx$$

$$= \frac{3}{256}\int_0^8 (8x^2 - x^3)\,dx$$

$$= \frac{3}{256}\left[\frac{8}{3}x^3 - \frac{1}{4}x^4\right]_0^8$$

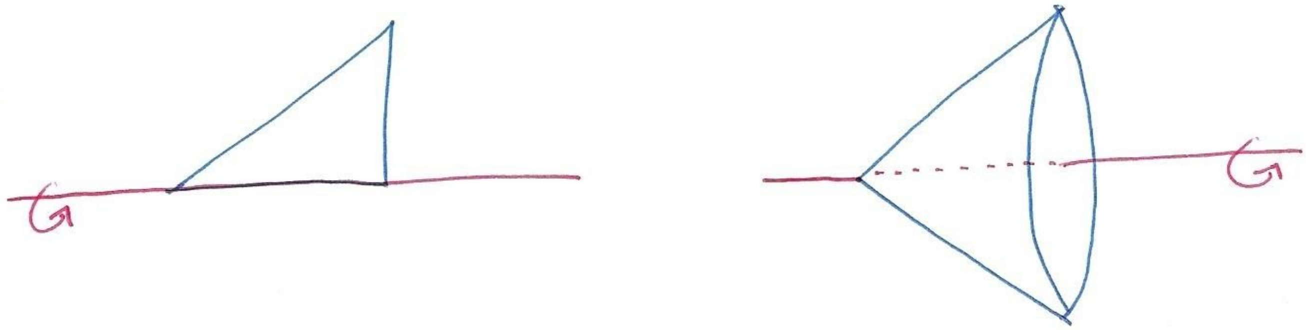
$$\boxed{=4}$$

Section 4.6: Volumes

Consider a plane shape, such as a rectangle. Suppose we have a line which does <u>not</u> pass through the plane shape, called the <u>axis of revolution</u>. Then we can revolve the plane shape around the axis of revolution to obtain a symmetric three-dimensional shape, called a <u>solid of revolution</u>.

More complicated plane shapes generate more complicated solids of revolution.

eg



Our goal is to find the volume of a solid of revolution.

Let's assume that we have a plane shape which lies under a curve $y=f(x)$ and above the x-axis on $[a,b]$ where the x-axis is also the axis of revolution. Then we can divide the plane shape into n rectangles, each of width Δx and height $f(x_i^*)$ where x_i^* is a sample point on each of the resulting subintervals.

When we rotate each rectangle around the axis of revolution, we obtain a <u>disc</u> of width Δx and radius $f(x_i^*)$. Then the volume of each disc is $\pi\left[f(x_i^*)\right]^2 \Delta x$ and so the volume V of the solid of revolution is given by

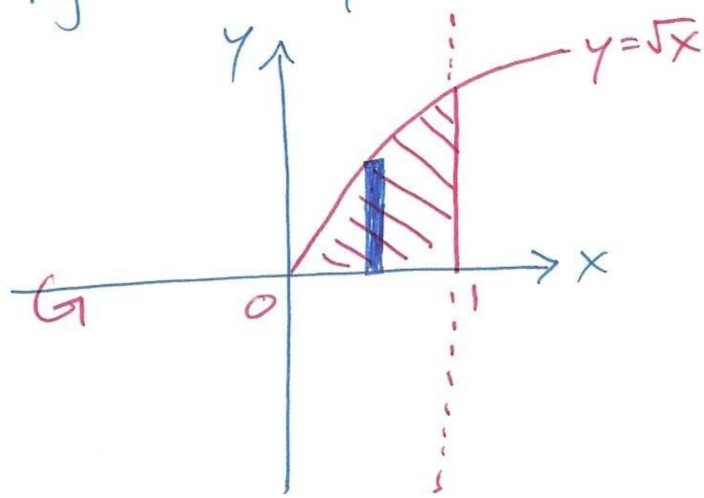
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi\left[f(x_i^*)\right]^2 \Delta x$$

$$= \pi \lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i^*)\right]^2 \Delta x$$

Thus

$$V = \pi \int_a^b [f(x)]^2 \, dx.$$

eg Find the volume of the solid obtained by revolving the region under $y=\sqrt{x}$ on $[0,1]$ around the x-axis.

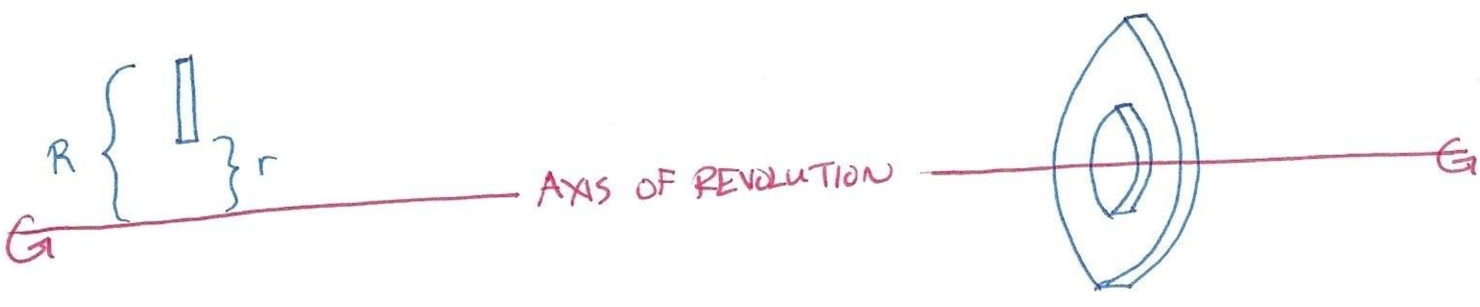


Here, $f(x)=\sqrt{x}$

$$[f(x)]^2 = [\sqrt{x}]^2 = x$$

so $V = \pi \int_0^1 x \, dx \boxed{= \frac{\pi}{2}}$

Now suppose that the axis of revolution is <u>not</u> a boundary curve of the plane shape.



In this situation, the solid generated by a rectangle is a <u>washer</u>, which is a large disc with a smaller disc removed from its centre.

Let R be the distance from the axis of revolution to the farther side of the rectangle. We call R the <u>outer radius</u> of the washer.

Let r be the distance from the axis of revolution to the closer side of the rectangle. We call r the <u>inner radius</u> of the washer.

Then the volume of the washer is given by

$$V = \pi R^2 - \pi r^2 = \pi (R^2 - r^2)$$

and hence the volume of the solid of revolution is

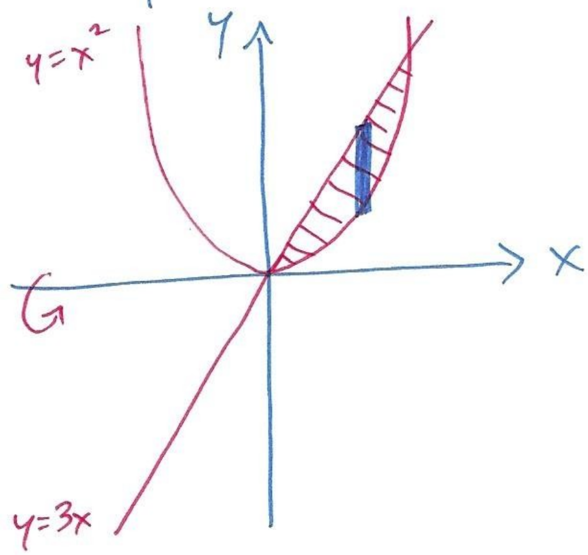
$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$$

In the case where the bottom boundary curve is the axis of revolution, $r(x) \equiv 0$ and so we have

$$V = \pi \int_a^b \left([R(x)]^2 - 0^2 \right) dx = \pi \int_a^b [R(x)]^2 \, dx$$

Thus we have now developed the <u>Disc - Washer Method</u> for volume.

eg Find the volume of the solid obtained by revolving around the x-axis the region bounded by $y = 3x$ and $y = x^2$.



Here, $R(x) = 3x$
$r(x) = x^2$

We set $x^2 = 3x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0$ and $x = 3$

$$V = \pi \int_0^3 \left([3x]^2 - [x^2]^2 \right) dx$$

$$= \pi \int_0^3 \left(9x^2 - x^4 \right) dx$$

$$\boxed{= \frac{162\pi}{5}}$$

—— THE END ——