Section 4.3

The constant harvesting model: $\frac{dy}{dt} = k(y-H)$, $y(0)=y_0$

This is a separable DE:

$$\frac{1}{y-H}\,dy = k\,dt$$

$$\int \frac{1}{y-H}\,dy = k\int dt$$

$$\ln|y-H| = kt + C$$

$$|y-H| = e^{kt+c} = e^{kt}\cdot e^{c}$$

$$y-H = Ce^{kt}$$

$$y = Ce^{kt} + H$$

Since $y(0)=y_0$, we have

$$y(0) = Ce^{0} + H = C+H = y_0$$

$$C = y_0 - H$$

The particular solution is $y = (y_0-H)e^{kt} + H$.

Hence the population survives if $y_0 - H > 0$ so $y_0 > H$, but perishes if $y_0 < H$.

Section 4.4: Predator - Prey Systems

Now suppose that the population $y(t)$ is a predator, and a second population $x(t)$ is its prey.

Assume that, in the absence of the predator population, the prey would grow exponentially, so $\frac{dx}{dt} = kx$ where the constant $k > 0$.

Assume that, in the absence of the prey population, the predators would decay exponentially, so $\frac{dy}{dt} = -ly$ where the constant $l > 0$.

Next we suppose that the rate of interaction between the predators and the prey is given by xy.

Then the prey population will decrease in proportion to xy,

so

$$\frac{dx}{dt} = kx - axy$$

for a constant $a > 0$.

Lastly, the predator population will increase in proportion to xy, so

$$\frac{dy}{dt} = -ly + bxy$$

for a constant $b > 0$.

What we have obtained is a system of coupled DEs known as the Lotka-Volterra equations:

$$\begin{cases} \dfrac{dx}{dt} = kx - axy \\ \dfrac{dy}{dt} = -ly + bxy \end{cases}$$

where a, b, k, l are positive constants.

To find any equilibrium points, we simultaneously set both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

First we set $\frac{dx}{dt} = 0$ so

$$kx - axy = 0$$
$$x(k - ay) = 0$$
$$\bar{x} = 0 \quad \text{or} \quad k - ay = 0$$
$$\bar{y} = \frac{k}{a}$$

Now we set $\frac{dy}{dt} = 0$ so $-ly + bxy = 0$.

If $\bar{x} = 0$ this becomes

$$-ly + 0 = 0$$
$$-ly = 0$$
$$\bar{y} = 0$$

so $(\bar{x}, \bar{y}) = (0, 0)$ is an equilibrium point.

For $\bar{y} = \frac{k}{a}$, then $-ly + bxy = 0$ becomes

$$-l \cdot \frac{k}{a} + bx \cdot \frac{k}{a} = 0$$

$$\frac{k}{a}(-l + bx) = 0$$

$$-l + bx = 0$$

$$\bar{x} = \frac{l}{b}$$

So another equilibrium point is $(\bar{x}, \bar{y}) = \left(\frac{l}{b}, \frac{k}{a}\right)$.

eg A population of foxes feeds on a colony of rabbits, such that their interactions can be described by the system of DEs

$$\begin{cases} \dfrac{dx}{dt} = 10x - \dfrac{1}{20}xy \\ \dfrac{dy}{dt} = -30y + \dfrac{1}{100}xy \end{cases}$$

Then $\bar{x} = \frac{l}{b} = \frac{30}{1/100} = 3000$

$$\bar{y} = \frac{k}{a} = \frac{10}{1/20} = 200$$

A population of 3000 rabbits can ideally sustain a population of 200 foxes.

Some populations co-operate to their mutual benefit. If we assume that both populations undergo exponential decay in the absence of the other, but that their rates of change are positively affected in proportion to the rate of their encounters xy, then we obtain the system of DEs

$$\begin{cases} \dfrac{dx}{dt} = -kx + axy \\[2ex] \dfrac{dy}{dt} = -ly + bxy \end{cases}$$

Alternatively, some populations are in competition with each other. If two populations would grow exponentially in the absence of the other, but their rates of change are negatively impacted in proportion to the rate of their encounters xy, then we have the system of DEs

$$\begin{cases} \dfrac{dx}{dt} = kx - axy \\[2ex] \dfrac{dy}{dt} = ly - bxy \end{cases}$$