Section 4.3

The logistic model is given by

$$\frac{dy}{dt} = k(1-my)y, \quad y(0) = y_0.$$

This is a separable DE:

$$\frac{1}{(1-my)y}\,dy = k\,dt$$

$$\int \frac{1}{(1-my)y}\,dy = k\int dt$$

We use the method of partial fractions for the integral on the left:

$$\frac{1}{(1-my)y} = \frac{A}{1-my} + \frac{B}{y}$$

$$1 = Ay + B(1-my)$$

For $y = \frac{1}{m}$ we have $1 = A\cdot\frac{1}{m} + B\cdot 0$

$$A = m$$

For $y=0$ we have $1 = A\cdot 0 + B\cdot 1$

$$B = 1$$

Thus $$\int \frac{1}{(1-my)y}\,dy = \int\left(\frac{m}{1-my} + \frac{1}{y}\right)dy$$

$$= m\cdot\frac{\ln|1-my|}{-m} + \ln|y| + C$$

$$= \ln|y| - \ln|1-my| + C$$

Then the general solution of the logistic model is given by

$$\int \frac{1}{(1-my)y}\,dy = k\int dt$$

$$\ln|y| - \ln|1-my| = kt + C$$

$$\ln\left|\frac{y}{1-my}\right| = kt + C$$

$$\left|\frac{y}{1-my}\right| = e^{kt+C} = e^{kt}\cdot e^{C}$$

$$\frac{y}{1-my} = Ce^{kt}$$

Since $y(0) = y_0$, we have

$$\frac{y_0}{1-my_0} = Ce^{0} = C$$

Next, we can write the solution of the logistic model explicitly by observing that

$$y = Ce^{kt}(1-my)$$

$$y = Ce^{kt} - Cme^{kt}y$$

$$y + Cme^{kt}y = Ce^{kt}$$

$$y(1+Cme^{kt}) = Ce^{kt}$$

$$y = \frac{Ce^{kt}}{1+Cme^{kt}}.$$

Let's consider the dynamics of the logistic model, and first suppose that $k > 0$. Then we have

$$\lim_{t \to \infty} y = \lim_{t \to \infty} \frac{Ce^{kt}}{1 + Cme^{kt}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\stackrel{\text{H}}{=} \lim_{t \to \infty} \frac{[Ce^{kt}]'}{[1 + Cme^{kt}]'}$$

$$= \lim_{t \to \infty} \frac{Cke^{kt}}{0 + Cmke^{kt}}$$

$$= \lim_{t \to \infty} \frac{1}{m} = \frac{1}{m}$$

Here, the constant $\frac{1}{m}$ is the <u>carrying capacity</u> or <u>saturation level</u>, which is the optimal population naturally supported by the environment.

If $k < 0$, we have

$$\lim_{t \to \infty} y = \lim_{t \to \infty} \frac{Ce^{kt}}{1 + Cme^{kt}} = \frac{0}{1 + 0} = 0$$

so the population tends towards extinction.

Def'n: An equilibrium point of a DE (also called a fixed point or a steady-state point) is any value of y for which $\frac{dy}{dt} = 0$. An equilibrium point is often denoted by $\bar{y}$.

eg For the logistic model, with $\frac{dy}{dt} = k(1-my)y$, we find any equilibrium points by setting

$$k(1-my)y = 0$$

$$1-my = 0 \quad \text{or} \quad y = 0$$

$$y = \frac{1}{m}$$

There are two equilibrium points: $\boxed{\bar{y} = 0 \text{ and } \bar{y} = \frac{1}{m}}$.

The only finite values to which the solutions of a DE can tend are those given by the equilibrium points.

eg For the natural growth model, with $\frac{dy}{dt} = ky$, we set

$$ky = 0 \rightarrow \bar{y} = 0$$

Because this model has only one equilibrium point, this is the only finite value that be achieved when we evaluate $\lim_{t \to \infty} y$.

A population is said to undergo <u>constant harvesting</u> when a constant amount h is regularly removed from the population. If we assume that, in the absence of harvesting, the population would follow the natural growth model, then this situation can be described by the model

$$\frac{dy}{dt} = ky - h, \quad y(0) = y_0.$$

Here we assume that $k > 0$ and $h > 0$.

To find any equilibrium points, we set

$$ky - h = 0$$

$$\bar{y} = \frac{h}{k} \quad \text{(only equilibrium point)}$$

Let $H = \frac{h}{k}$. Then we can rewrite the DE as

$$\frac{dy}{dt} = ky - h = k(y - h/k)$$

$$= k(y - H).$$