# Section 4.2

The exponential growth/decay formula: $y = y_0 e^{kt}$

eg A colony of 100 rabbits is brought to a large island with abundant food and no predators, such that its rate of change is proportional to its current size. After 2 years, there are 250 rabbits. Determine the size of the population after 6 years.

Here $y_0 = y(0) = 100$ and $y(2) = 250$.

Thus $y = 100e^{kt}$ and so

$$y(2) = 100e^{k \cdot 2} = 250$$

$$e^{2k} = \frac{250}{100} = \frac{5}{2}$$

$$\ln(e^{2k}) = \ln(5/2)$$

$$2k = \ln(5/2)$$

$$k = \frac{1}{2}\ln(5/2)$$

Now we have $y = 100e^{\frac{1}{2}\ln(5/2)t}$

$$y(6) = 100e^{\frac{1}{2}\ln(5/2)\cdot 6}$$

$$= 100e^{3\ln(5/2)}$$

$$= 100e^{\ln((5/2)^3)}$$

$$= 100e^{\ln(125/8)}$$

$$= 100 \cdot \frac{125}{8} \approx 1563$$

After 6 years, there are about <u>1563 rabbits</u>.

eg A patient has a bacterial infection. A course of antibiotics causes the bacteria to undergo exponential decay, such that after 4 days two-fifths of the bacteria have been eliminated. Find the fraction of the original bacteria population that remains after 8 days.

If $y_0$ is the initial population then we are given that $y(4) = y_0 - \frac{2}{5}y_0 = \frac{3}{5}y_0$. This means that

$$y = y_0 e^{kt}$$

$$\frac{y(4) = y_0 e^{k \cdot 4}}{y_0} = \frac{\frac{3}{5}y_0}{y_0}$$

$$e^{4k} = \frac{3}{5}$$

$$\ln(e^{4k}) = \ln(3/5)$$

$$4k = \ln(3/5)$$

$$k = \frac{1}{4}\ln(3/5)$$

Then $y = y_0 e^{\frac{1}{4}\ln(3/5)t}$

$$y(8) = y_0 e^{\frac{1}{4}\ln(3/5)\cdot 8}$$

$$= y_0 e^{2\ln(3/5)}$$

$$= y_0 e^{\ln(9/25)}$$

$$= y_0 \cdot \frac{9}{25}$$

Hence $\frac{9}{25}$ of the original bacteria remains after 8 days.

eg Radioactive elements decay spontaneously at a rate proportional to the mass of the sample. Hence they undergo exponential decay. The <u>half-life</u> measures the time it takes for half the sample to decay. Carbon-14 has a half-life of 5730 years. Determine, to the nearest year, how long it takes for a sample of 100 mg of Carbon-14 to decay to 10 mg.

We have $y_0 = 100$ and we know that $y(5730) = 50$.

Then $y = y_0 e^{kt}$

$$= 100e^{kt}$$

$$y(5730) = 100e^{k \cdot 5730} = 50$$

$$e^{5730k} = \frac{1}{2}$$

$$\ln(e^{5730k}) = \ln(1/2) = \ln(2^{-1})$$

$$5730k = -\ln(2)$$

$$k = \frac{-\ln(2)}{5730}$$

Now we have $y = 100e^{-\frac{\ln(2)}{5730}t}$ so we set

$$10 = 100e^{-\frac{\ln(2)}{5730}t}$$

$$\frac{1}{10} = e^{-\frac{\ln(2)}{5730}t}$$

$$-\ln(10) = -\frac{\ln(2)}{5730}t$$

$$t = \frac{5730\ln(10)}{\ln(2)} \approx 19036$$

It takes about 19,036 years for the sample to decay to 10 mg.

# Section 4.3: Models for Population Growth

The mathematical model described by $\frac{dy}{dt} = ky$ is sometimes called the natural growth model.

Here $k$ must represent the key natural factors that influence population growth. One such factor is the virility $v$ of the population, and another is the food supply $f$. If these factors are of equal importance then we set $k = vf$.

More realistically, we could account for the impact on the food supply of the population. Suppose $c$ is the rate of consumption. Then the food supply would instead be represented $f - cy$.

Now we have a new mathematical model given by the DE

$$\begin{aligned} \frac{dy}{dt} &= v(f - cy)y \\ &= vf\left(1 - \frac{c}{f}y\right)y \\ &= k(1 - my)y \end{aligned}$$

where $k = vf$ and $m = \frac{c}{f}$. This is the logistic model.