Section 1.1

eg $\int (2x-3)^{60}\,dx$

Compare: $\int x^{60}\,dx = \dfrac{x^{61}}{61} + C$

Guess: $\int (2x-3)^{60}\,dx = \dfrac{(2x-3)^{61}}{61} + C$

Check by differentiating:

$$\left[\frac{(2x-3)^{61}}{61} + C\right]' = \frac{1}{61}\cdot 61(2x-3)^{60}\cdot 2 + 0$$

$$= 2(2x-3)^{60}$$

$$\neq (2x-3)^{60}$$

So our guess is <u>incorrect</u>. However, we can write

$$\int 2(2x-3)^{60}\,dx = \frac{(2x-3)^{61}}{61} + C$$

$$2\int (2x-3)^{60}\,dx = \frac{(2x-3)^{61}}{61} + C$$

$$\int (2x-3)^{60}\,dx = \frac{(2x-3)^{61}}{2\cdot 61} + C$$

$$\boxed{= \frac{(2x-3)^{61}}{122} + C}$$

eg $\int \sin(7x)\,dx$

Compare: $\int \sin(x)\,dx = -\cos(x) + C$

Guess: $\int \sin(7x)\,dx = -\cos(7x) + C$

Check: $[-\cos(7x) + C]' = \sin(7x)\cdot 7 + 0$

$= 7\sin(7x)$

Thus $\int 7\sin(7x)\,dx = -\cos(7x) + C$

$$\boxed{\int \sin(7x)\,dx = -\frac{1}{7}\cos(7x) + C}$$

<u>Theorem</u>: If $f(x)$ is a function with antiderivative $F(x)$ then

$$\int f(mx+b)\,dx = \frac{F(mx+b)}{m} + C.$$

eg $\int \frac{1}{(2-x)^6}\,dx$ Compare: $\int \frac{1}{x^6}\,dx$

$= \int x^{-6}\,dx = \frac{x^{-5}}{-5} + C$

Then $\int \frac{1}{(2-x)^6}\,dx = \frac{(2-x)^{-5}}{(-5)\cdot(-1)} + C$

$$\boxed{\begin{aligned} &= \frac{1}{5}(2-x)^{-5} + C \\ &= \frac{1}{5(2-x)^5} + C \end{aligned}}$$

eg $\int \sec^2\left(\frac{4x+6}{3}\right) dx$ Compare: $\int \sec^2(x)\,dx$

$= \tan(x) + C$

$= \int \sec^2\left(\frac{4}{3}x + 2\right) dx$

$= \dfrac{\tan\left(\frac{4}{3}x + 2\right)}{4/3} + C$

$\boxed{= \frac{3}{4}\tan\left(\frac{4}{3}x + 2\right) + C}$

eg $\int \frac{1}{\frac{1}{4}x - 3}\, dx$ Compare: $\int \frac{1}{x}\, dx$

$= \ln|x| + C$

$= \dfrac{\ln\left|\frac{1}{4}x - 3\right|}{1/4} + C$

$\boxed{= 4\ln\left|\frac{1}{4}x - 3\right| + C}$

Alternatively, we could first rewrite:

$$\int \frac{1}{\frac{1}{4}x - 3}\, dx = \int \frac{4}{x - 12}\, dx$$

$$= 4\int \frac{1}{x-12}\, dx$$

$$= 4\,\frac{\ln|x-12|}{1} + C$$

$$\boxed{= 4\ln|x-12| + C}$$

But we can write

$$4 \ln \left|\tfrac{1}{4}x-3\right| + C = 4\ln\left|\tfrac{1}{4}(x-12)\right| + C$$
$$= 4\left(\ln\left|\tfrac{1}{4}\right| + \ln|x-12|\right) + C$$
$$= 4\ln|x-12| + 4\ln\left|\tfrac{1}{4}\right| + C$$
$$= 4\ln|x-12| + C$$

Could we obtain a similar result for quadratic composition instead of linear composition? That is, for an integral of the form

$$\int f(ax^2+bx+c)\,dx$$

eg $\int (2x^2-3)^{60}\,dx$ Compare: $\int x^{60}\,dx = \frac{x^{61}}{61} + C$

Guess: $\int (2x^2-3)^{60}\,dx = \dfrac{(2x^2-3)^{61}}{61} + C$

Check: $\left[\dfrac{(2x^2-3)^{61}}{61} + C\right]' = \dfrac{61(2x^2-3)^{60}\cdot 4x}{61} + 0$

$$= 4x(2x^2-3)^{60}$$

Thus $\int 4x(2x^2-3)^{60}\,dx = \dfrac{(2x^2-3)^{61}}{61} + C$

but we have no way to obtain $\int (2x^2-3)^{60}\,dx$.

Sadly, no simple pattern exists beyond linear composition.

Note that

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

eg $\int 2x\,dx = x^2 + C$

$$\frac{d}{dx}\left[\int 2x\,dx\right] = \frac{d}{dx}\left[x^2 + C\right] = 2x + 0 \boxed{= 2x}$$

However,

$$\int f'(x)\,dx = f(x) + C.$$

eg $[x^2]' = 2x$

$$\int [x^2]'\,dx = \int 2x\,dx \boxed{= x^2 + C}$$