Section 4.2: Separable Equations

A DE is <u>separable</u> if it can be written such that all expressions involving the unknown function y appear only on one side of the equation, and all expressions involving the independent variable t appear only on the other side. To do this we use the relationship between differentials given by

$$dy = \frac{dy}{dt}\,dt \quad \text{or} \quad dy = y'\,dt.$$

Then we integrate both sides to solve the DE.

eg $$t^4 \frac{dy}{dt} - \frac{1}{y^2} = 0$$

We can rewrite this DE as

$$t^4 \frac{dy}{dt} = \frac{1}{y^2}$$

$$y^2 \frac{dy}{dt} = t^{-4}$$

$$y^2\,dy = t^{-4}\,dt$$

Since we have successfully separated the variables y and t, this is a separable DE.

Now we integrate both sides:

$$\int y^2\,dy = \int t^{-4}\,dt$$

$$\frac{y^3}{3} = \frac{t^{-3}}{-3} + C$$

$$y^3 = -\frac{1}{t^3} + C$$

$$\boxed{y = \sqrt[3]{C - \frac{1}{t^3}}}$$

We are often unable to express the solution of a separable DE as an explicit function. Instead, it is common to write these solutions in an implicit form.

eg Solve the IVP

$$ty(y^2+1)\frac{dy}{dt} = 7, \quad y(1)=0$$

This is a separable DE:

$$y(y^2+1)\,dy = \frac{7}{t}\,dt$$

$$\int (y^3+y)\,dy = 7\int \frac{1}{t}\,dt$$

$$\frac{y^4}{4} + \frac{y^2}{2} = 7\ln|t| + C \quad \text{(general solution)}$$

$$y^4 + 2y^2 = 28\ln|t| + C$$

$$y^4+2y^2-28\ln|t| = C$$

Since $y(1)=0$, this becomes $0+0-28\ln(1) = C$ so $C=0$. Hence the particular solution is

$$\boxed{y^4 + 2y^2 - 28\ln|t| = 0}.$$

eg Recall the DE $\frac{dy}{dt} = ky$. Observe that this is a separable DE because it can be written

$$\frac{1}{y}\,dy = k\,dt$$

$$\int \frac{1}{y}\,dy = k\int dt$$

$$\ln|y| = kt + C$$

We can assume that $y > 0$, so the general solution becomes

$$\ln(y) = kt + C$$

$$e^{\ln(y)} = e^{kt+C}$$

$$y = e^{kt} \cdot e^{C} = Ce^{kt}$$

Further assume that we have the initial condition given by $y(0) = y_0$. Now we know that

$$y(0) = Ce^{0} = C = y_0$$

so the particular solution is given by

$$y = y_0 e^{kt}.$$

Often we are interested in using the solutions of a DE to help us understand the overall behaviour of the model, called its <u>dynamics</u>. In particular, we often want to know what happens as $t \to \infty$.

To explore the dynamics of the DE $\frac{dy}{dt} = ky$, we can consider the limit as $t \to \infty$ of its solution.

If $k > 0$ we have

$$\lim_{t\to\infty} y = \lim_{t\to\infty} y_0 e^{kt} = \infty$$

which is <u>exponential growth</u>.

If $k < 0$ then

$$\lim_{t\to\infty} y = \lim_{t\to\infty} y_0 e^{kt} = 0$$

which is <u>exponential decay</u>.

If $k = 0$ then the solution of the DE is

$$y = y_0 e^{0 \cdot t} = y_0$$

$$\lim_{t\to\infty} y = \lim_{t\to\infty} y_0 = y_0$$

We can also determine these dynamics directly from the DE.

If $k > 0$, then $\frac{dy}{dt} = ky > 0$ so y is always increasing.

If $k < 0$, then $\frac{dy}{dt} = ky < 0$ so y is always decreasing.

If $k = 0$, then $\frac{dy}{dt} = ky = 0$ so y does not change.

This is <u>qualitative analysis</u> of the DE.