

Section 4.1

eg Find the particular solution of the IVP

$$t^2 \frac{dy}{dt} - 3t^4 + 1 = 0, \quad y(1) = 0.$$

First we find the general solution:

$$t^2 \frac{dy}{dt} = 3t^4 - 1$$

$$\frac{dy}{dt} = 3t^2 - t^{-2}$$

$$\begin{aligned} y(t) &= \int (3t^2 - t^{-2}) dt \\ &= 3 \cdot \frac{t^3}{3} - \frac{t^{-1}}{-1} + C \\ &= t^3 + \frac{1}{t} + C \end{aligned}$$

We are given that $y=0$ when $t=1$, so we substitute into the general solution to get

$$0 = 1^3 + \frac{1}{1} + C$$

$$0 = 2 + C \rightarrow C = -2$$

Therefore the particular solution is

$$\boxed{y(t) = t^3 + \frac{1}{t} - 2}.$$

We often apply DEs to the study of objects in motion or kinematics. We assume that an object moves such that its position is given by a function $s(t)$, its velocity is $v(t)$, and its acceleration is $a(t)$. We know that

$$s'(t) = v(t) \quad \text{and} \quad v'(t) = a(t).$$

Thus we can also write

$$s(t) = \int v(t) dt \quad \text{and} \quad v(t) = \int a(t) dt.$$

eg A ball is thrown upwards by a girl standing on a cliff that is 50 m high. She throws it with a velocity of 28 m/sec. To two decimal places, determine the time at which the ball reaches its maximum height, and measure that height.

We assume that gravity is the only force acting on the ball, so $a(t) = -9.8$. We are given that $s(0) = 50$ and $v(0) = 28$.

Then

$$v(t) = \int (-9.8) dt$$
$$= -9.8t + C$$

Since $v(0) = 28$, we have

$$28 = -9.8 \cdot 0 + C$$
$$28 = C$$

and hence the particular solution is $v(t) = -9.8t + 28$.

Next,

$$\begin{aligned}s(t) &= \int (-9.8t + 28) dt \\ &= -9.8 \cdot \frac{t^2}{2} + 28t + D \\ &= -4.9t^2 + 28t + D\end{aligned}$$

Since $s(0) = 50$, we obtain $50 = -4.9 \cdot 0 + 28 \cdot 0 + D$

$$50 = D$$

Thus the particular solution is $s(t) = -4.9t^2 + 28t + 50$.

Now we set $v(t) = 0$ so

$$-9.8t + 28 = 0$$

$$t = \frac{28}{9.8} \approx 2.86$$

The ball reaches its maximum height at about 2.86 sec.

$$\begin{aligned}\text{Finally, } s(2.86) &= -4.9 \cdot (2.86)^2 + 28 \cdot (2.86) + 50 \\ &= 90\end{aligned}$$

so the maximum height achieved by the ball is 90 m.

DEs may involve derivatives of the unknown function of any order. For instance, we can write

$$\frac{d^2s}{dt^2} = a(t)$$

This is a second-order DE.

The order of a DE is the order of the highest derivative which appears in the equation.

eg $\frac{dy}{dt} = ky$ is a first-order DE

eg Solve the IVP

$$f''(t) - \cos\left(\frac{1}{2}t\right) = 0$$

given initial conditions $f(0) = 0$ and $f(\pi) = 0$.

We have

$$f''(t) = \cos\left(\frac{1}{2}t\right)$$

$$f'(t) = \int \cos\left(\frac{1}{2}t\right) dt$$
$$= \frac{\sin\left(\frac{1}{2}t\right)}{\frac{1}{2}} + C_1$$

$$= 2\sin\left(\frac{1}{2}t\right) + C_1$$

$$f(t) = \int (2\sin\left(\frac{1}{2}t\right) + C_1) dt$$
$$= 2 \cdot \frac{-\cos\left(\frac{1}{2}t\right)}{\frac{1}{2}} + C_1 t + C_2$$
$$= -4\cos\left(\frac{1}{2}t\right) + C_1 t + C_2$$

Since $f(0) = 0$, we have

$$0 = -4\cos(0) + C_1 \cdot 0 + C_2$$

$$0 = -4 + C_2$$

$$C_2 = 4$$

Now we have

$$f(t) = -4 \cos\left(\frac{1}{2}t\right) + C_1 t + 4$$

Since $f(\pi) = 0$, we have

$$0 = -4 \cos\left(\frac{\pi}{2}\right) + C_1 \cdot \pi + 4$$

$$0 = \pi C_1 + 4$$

$$C_1 = -\frac{4}{\pi}$$

The particular solution is $f(t) = -4 \cos\left(\frac{1}{2}t\right) - \frac{4}{\pi}t + 4$.

Higher-order DEs also appear in many applications.

eg Hooke's Law states that the force exerted by a spring to return to its equilibrium state is proportional to its displacement $s(t)$. Since force is the product of mass and acceleration, Hooke's Law can be expressed as the second-order DE

$$m \frac{d^2 s}{dt^2} = ks$$

for some constant k .