Section 3.4

We often need to use l'Hôpital's Rule to evaluate the limit associated with an improper integral.

eg $\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx$

Since $x=0$ is a discontinuity of $\frac{\ln(x)}{\sqrt{x}}$, we write

$$\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx = \lim_{T\to 0^+} \int_T^1 \frac{\ln(x)}{\sqrt{x}}\,dx$$

We try integration by parts with

$w = \ln(x)$ $\qquad$ $dw = \frac{1}{x}\,dx$

$dv = x^{-1/2}\,dx$ $\qquad$ $v = 2\sqrt{x}$

Now we can write

$$\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx = \lim_{T\to 0^+}\left[\left[2\sqrt{x}\ln(x)\right]_T^1 - \int_T^1 2\sqrt{x}\cdot\frac{1}{x}\,dx\right]$$

$$= \lim_{T\to 0^+}\left[\left[2\sqrt{x}\ln(x)\right]_T^1 - 2\int_T^1 x^{-1/2}\,dx\right]$$

$$= \lim_{T\to 0^+}\left[2\sqrt{x}\ln(x) - 2\cdot\frac{x^{1/2}}{1/2}\right]_T^1$$

$$= \lim_{T\to 0^+}\left[2\sqrt{x}\ln(x) - 4\sqrt{x}\right]_T^1$$

$$= \lim_{T\to 0^+}\left[(2\cdot 1\cdot 0 - 4\cdot 1) - (2\sqrt{T}\ln(T) - 4\sqrt{T})\right]$$

$$= \lim_{T\to 0^+}\left[-4 - 2\sqrt{T}\ln(T) + 4\sqrt{T}\right]$$

$$= -4 - 2\lim_{T\to 0^+}\sqrt{T}\ln(T)$$

$$= -4 - 2\lim_{T\to 0^+}\frac{\ln(T)}{1/\sqrt{T}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$\overset{H}{=} -4 - 2\lim_{T\to 0^+}\frac{\frac{d}{dT}[\ln(T)]}{\frac{d}{dT}[T^{-1/2}]}$$

$$= -4 - 2\lim_{T\to 0^+}\frac{1/T}{-\frac{1}{2}T^{-3/2}}$$

$$= -4 - 2\lim_{T\to 0^+}(-2\sqrt{T})$$

$$= -4 + 4\lim_{T\to 0^+}\sqrt{T}$$

$$= -4 + 0 \; \boxed{= -4} \quad (\text{convergent})$$

Section 4.1: Modelling with Differential Equations

In real-world applications, we often have information about the rate of change of a process, and our goal is to determine information about the function which describes the process itself.

eg A simple way to model the growth of a population is to assume that its rate of change is proportional to its current size. If $y(t)$ represents the size of the population at time t, we could write

$$\frac{dy}{dt} = ky$$

for some constant k.

A <u>differential equation</u> (DE) is an equation which involves an unknown function and one or more of its derivatives.

We want to identify functions which satisfy the DE, so that the two sides of the DE are the same when y is substituted into the equation.

eg Given the DE $\frac{dy}{dt} = ky$, determine whether $y(t) = 5e^{kt}$ and $y(t) = 5\sin(kt)$ are solutions of the DE.

For $y(t) = 5e^{kt}$, we have

$$\frac{dy}{dt} = \frac{d}{dt}\left[5e^{kt}\right] = 5 \cdot e^{kt} \cdot k = 5ke^{kt}$$

$$ky = k \cdot 5e^{kt} = 5ke^{kt}$$

Since these are equal, $y(t) = 5e^{kt}$ <u>is</u> a solution of the DE.

For $y(t) = 5\sin(kt)$ we have

$$\frac{dy}{dt} = \frac{d}{dt}\left[5\sin(kt)\right] = 5\cos(kt) \cdot k = 5k\cos(kt)$$

$$ky = k \cdot 5\sin(kt) = 5k\sin(kt)$$

These are not equal, so $y(t) = 5\sin(kt)$ is <u>not</u> a solution of the DE.

Typically, a DE will be satisfied by a family of functions called its <u>general solution</u>. It is often found through integration.

The simplest type of DE is one in which the unknown function appears only in the form of its derivative.

eg Consider the DE $\frac{dy}{dt} - 2t^3 = 3$.

We can rewrite the DE as

$$\frac{dy}{dt} = 2t^3 + 3$$

so

$$y(t) = \int \frac{dy}{dt}\,dt = \int (2t^3+3)\,dt$$
$$= 2\cdot\frac{t^4}{4} + 3t + C$$
$$= \frac{1}{2}t^4 + 3t + C$$

The general solution is $\boxed{y(t) = \frac{1}{2}t^4 + 3t + C}$.

In practice, we often have an additional piece of information about the unknown function, called an <u>initial condition</u>.

A specific function which satisfies both the DE and the initial condition is called the <u>particular solution</u>.

The combination of a DE and an initial condition is called an <u>initial value problem</u> (IVP).