Section 3.4

Now suppose that we have an integral of the form $\int_{-\infty}^{b} f(x)dx$.

Then we define

$$\int_{-\infty}^{b} f(x)dx = \lim_{T\to-\infty} \int_{T}^{b} f(x)dx.$$

eg

$$\int_{-\infty}^{1} e^{2x}\,dx = \lim_{T\to-\infty} \int_{T}^{1} e^{2x}\,dx$$

$$= \lim_{T\to-\infty} \left[\frac{1}{2}e^{2x}\right]_{T}^{1}$$

$$= \lim_{T\to-\infty} \left[\frac{1}{2}e^{2} - \frac{1}{2}e^{2T}\right]$$

$$= \frac{1}{2}e^{2} - 0 \boxed{= \frac{1}{2}e^{2}} \quad \text{(convergent)}$$

Finally, suppose we have an integral of the form $\int_{-\infty}^{\infty} f(x)\,dx$.

We choose an appropriate value p and use the Additive Interval Property to write

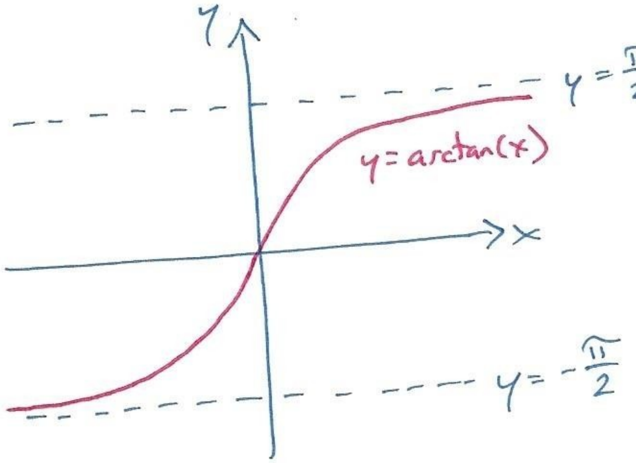
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{p} f(x)dx + \int_{p}^{\infty} f(x)dx.$$

eg

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

First we have

$$\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx = \lim_{T\to-\infty} \int_{T}^{0} \frac{1}{x^2+1}\,dx$$



$$= \lim_{T\to-\infty} \Big[\arctan(x)\Big]_{T}^{0}$$

$$= \lim_{T\to-\infty} \Big[0 - \arctan(T)\Big]$$

$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Next we have

$$\int_{0}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{T\to\infty} \int_{0}^{T} \frac{1}{x^2+1}\,dx$$

$$= \lim_{T\to\infty} \Big[\arctan(x)\Big]_{0}^{T}$$

$$= \lim_{T\to\infty} \Big[\arctan(T) - 0\Big]$$

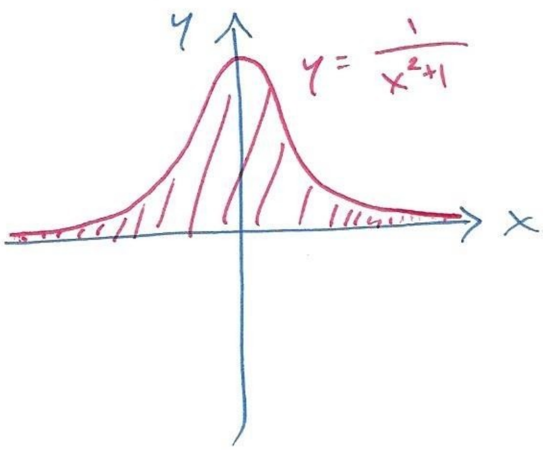
$$= \frac{\pi}{2}$$

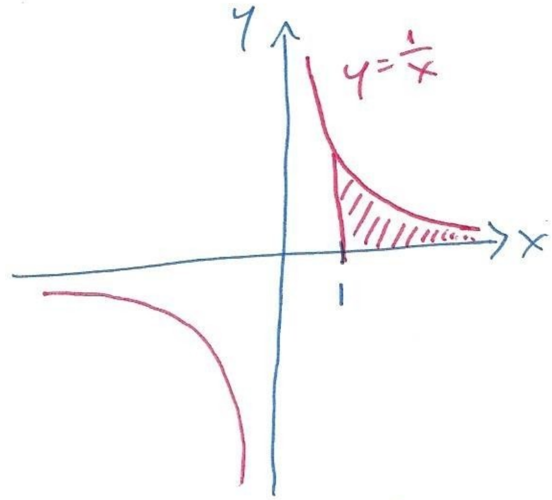
Thus $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2} + \frac{\pi}{2} \boxed{= \pi}$ (convergent)

As long as $f(x) \geq 0$, we can still interpret $\int_a^b f(x)\,dx$ as representing the area under $y = f(x)$ on $[a, b]$ even when the integral is improper.

Some regions of infinite extent can have a finite area.

eg The region under $y = \frac{1}{x^2+1}$ on $(-\infty, \infty)$ has $A = \pi$.





Other regions of infinite extent can have an infinite area,

eg We showed that $\int_1^\infty \frac{1}{x}\, dx$ is divergent so no finite value can be assigned to the area under $y = \frac{1}{x}$ on the interval $[1, \infty)$.

We can use any appropriate integration technique to evaluate an improper integral.

eg $$\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx = \lim_{T\to\infty}\int_0^T \frac{2x}{(x^2+1)^2}\,dx$$

We let $u = x^2+1$ so $du = 2x\,dx$

When $x=0$, $u=1$

$x=T$, $u=T^2+1$

The integral becomes

$$\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx = \lim_{T\to\infty}\int_1^{T^2+1} u^{-2}\,du$$

$$= \lim_{T\to\infty}\left[\frac{u^{-1}}{-1}\right]_1^{T^2+1}$$

$$= \lim_{T\to\infty}\left[\frac{-1}{T^2+1}+1\right]$$

$$= 0+1 \;\boxed{=1}$$ (convergent)

For the second type of improper integral, let's first consider the case where we have $\int_a^b f(x)\,dx$ where $f(x)$ is discontinuous at the lower bound $x=a$. We write

$$\int_a^b f(x)\,dx = \lim_{T\to a^+}\int_T^b f(x)\,dx$$

eg $\int_0^4 \frac{1}{\sqrt{x}}\,dx$

Note that $\frac{1}{\sqrt{x}}$ is discontinuous at $x=0$, so we write

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{T\to 0^+} \int_T^4 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{T\to 0^+} \left[\frac{x^{1/2}}{1/2}\right]_T^4$$

$$= 2\lim_{T\to 0^+} \left[2 - T^{1/2}\right]$$

$$= 2\cdot(2-0) \boxed{=4} \quad \text{(convergent)}$$

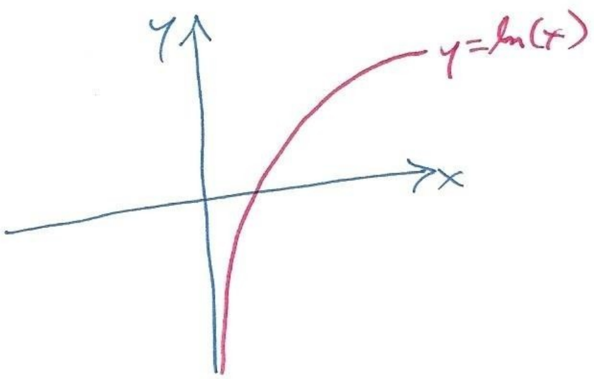
If $f(x)$ is discontinuous at the upper bound $x=b$, we write

$$\int_a^b f(x)\,dx = \lim_{T\to b^-} \int_a^T f(x)\,dx.$$

eg $\int_{-2}^{2} \frac{1}{x-2}\,dx$

Note that $\frac{1}{x-2}$ is discontinuous at $x=2$, so we have

$$\int_{-2}^{2} \frac{1}{x-2}\,dx = \lim_{T\to 2^-} \int_{-2}^{T} \frac{1}{x-2}\,dx$$



$$= \lim_{T\to 2^-} \Big[\ln|x-2|\Big]_{-2}^{T}$$

$$= \lim_{T\to 2^-} \Big[\ln|T-2| - \ln(4)\Big]$$

$$= -\infty$$

This improper integral is <u>divergent</u>.

If $f(x)$ is discontinuous at $x=p$ where $a<p<b$ then we apply the Additive Interval Property to write

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$$

eg $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

We note that $\dfrac{1}{\sqrt[3]{x-1}}$ is discontinuous at $x=1$, so we write

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$= -\frac{3}{2} + 6 \boxed{= \frac{9}{2}}$$ (convergent)