Section 3.3

The method of trigonometric substitution often leads to the use of trigonometric integral strategies.

eg $\int \sqrt{4-x^2}\,dx$

We let $x = 2\sin(\theta)$ so $dx = 2\cos(\theta)\,d\theta$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2(\theta)}$$
$$= \sqrt{4\cos^2(\theta)}$$
$$= 2\cos(\theta)$$

The integral becomes

$$\int \sqrt{4-x^2}\,dx = \int 2\cos(\theta)\cdot 2\cos(\theta)\,d\theta$$
$$= 4\int \cos^2(\theta)\,d\theta$$
$$= 4\int \frac{1+\cos(2\theta)}{2}\,d\theta$$
$$= 2\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$
$$= 2\theta + \sin(2\theta) + C$$
$$= 2\arcsin\left(\tfrac{1}{2}x\right) + 2\sin(\theta)\cos(\theta) + C$$
$$= 2\arcsin\left(\tfrac{1}{2}x\right) + 2\cdot\tfrac{1}{2}x\cdot\tfrac{1}{2}\sqrt{4-x^2} + C$$
$$\boxed{= 2\arcsin\left(\tfrac{1}{2}x\right) + \tfrac{1}{2}x\sqrt{4-x^2} + C}$$

As with u-substitution, in order to apply the method of trigonometric substitution to a definite integral, the bounds must also be transformed into the corresponding values of the new variable θ. However, there is also no need to transform back into the original variable after integrating: we simply apply FTC ② in terms of θ.

eg $\int_0^{\sqrt{3}} \frac{1}{(x^2+9)^{3/2}}\,dx$

Note that $\int_0^{\sqrt{3}} \frac{1}{(x^2+9)^{3/2}}\,dx = \int_0^{\sqrt{3}} \frac{1}{\left(\sqrt{x^2+9}\right)^3}\,dx$

We let $x = 3\tan(\theta)$ so $dx = 3\sec^2(\theta)\,d\theta$

$$\sqrt{x^2+9} = \sqrt{9\tan^2(\theta)+9} = 3\sec(\theta)$$

$$\left(\sqrt{x^2+9}\right)^3 = \left[3\sec(\theta)\right]^3 = 27\sec^3(\theta)$$

When $x=0$, we have $3\tan(\theta) = 0$

$$\tan(\theta) = 0$$

$$\theta = \arctan(0) = 0$$

When $x = \sqrt{3}$, we have $3\tan(\theta) = \sqrt{3}$

$$\tan(\theta) = \frac{\sqrt{3}}{3}$$

$$\theta = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

The integral becomes

$$\int_0^{\sqrt{3}} \frac{1}{(x^2+9)^{3/2}}\,dx = \int_0^{\pi/6} \frac{1}{27\sec^3(\theta)} \cdot 3\sec^2(\theta)\,d\theta$$

Now we have

$$\int_0^{\sqrt{3}} \frac{1}{(x^2+9)^{3/2}}\,dx = \frac{1}{9}\int_0^{\pi/6} \frac{1}{\sec(\theta)}\,d\theta$$

$$= \frac{1}{9}\int_0^{\pi/6} \cos(\theta)\,d\theta$$

$$= \frac{1}{9}\Big[\sin(\theta)\Big]_0^{\pi/6}$$

$$= \frac{1}{9}\left[\sin\left(\frac{\pi}{6}\right) - \sin(0)\right]$$

$$= \frac{1}{9}\left[\frac{1}{2} - 0\right]$$

$$\boxed{= \frac{1}{18}}$$

Section 3.4: Improper Integrals

Given a definite integral $\int_a^b f(x)\,dx$, we say that it is <u>improper</u> if it does not fulfill the requirements of FTC ②. There are two ways this can happen:

① The interval of integration could be infinite or semi-infinite. That is, b could be ∞, a could be $-\infty$, or both.

② The integrand $f(x)$ could be discontinuous on the interval of integration.

We want to determine whether it's possible to evaluate each kind of improper integral and, if so, how.

<u>Case ①</u>: First, suppose we have a definite integral of the form $\int_a^\infty f(x)\,dx$. We define

$$\int_a^\infty f(x)\,dx = \lim_{T\to\infty} \int_a^T f(x)\,dx$$

So that FTC ② can be applied to the definite integral involving the <u>dummy variable</u> T, since it can be taken to represent an arbitrary real number. If the limit exists, we say that the improper integral is <u>convergent</u>. Otherwise, it is <u>divergent</u>.

eg $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{T\to\infty} \int_1^T \frac{1}{x^2}\,dx$

$$= \lim_{T\to\infty} \left[-\frac{1}{x}\right]_1^T$$

$$= \lim_{T\to\infty} \left[-\frac{1}{T} + 1\right] = 0 + 1 \boxed{= 1}$$

This improper integral is <u>convergent</u>.

eg $\int_1^\infty \frac{1}{x}\,dx = \lim_{T\to\infty} \int_1^T \frac{1}{x}\,dx$

$$= \lim_{T\to\infty} \left[\ln|x|\right]_1^T$$

$$= \lim_{T\to\infty} \left[\ln|T| - 0\right] = \infty$$

This improper integral is <u>divergent</u>.

eg $\int_0^\infty \cos(x)\,dx = \lim_{T\to\infty} \int_0^T \cos(x)\,dx$

$$= \lim_{T\to\infty} \left[\sin(x)\right]_0^T$$

$$= \lim_{T\to\infty} \left[\sin(T) - 0\right]$$ which does <u>not</u> exist

This improper integral <u>diverges</u>.