Section 3.3

When the integral involves the form $\sqrt{x^2+k^2}$, we use the substitution $x = k\tan(\theta)$. Then

$$\sqrt{x^2+k^2} = \sqrt{k^2\tan^2(\theta)+k^2} = \sqrt{k^2[\tan^2(\theta)+1]}$$

$$= \sqrt{k^2\sec^2(\theta)} = k\sec(\theta)$$

This also means that $dx = \frac{d}{d\theta}[k\tan(\theta)]\,d\theta$

$$= k\sec^2(\theta)\,d\theta.$$

eg $\int \dfrac{1}{\sqrt{x^2+1}}\,dx$

Here $k=1$ so we set $x = \tan(\theta)$ so $dx = \sec^2(\theta)\,d\theta$.

Furthermore $\sqrt{x^2+1} = \sec(\theta)$.

The integral becomes

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sec(\theta)}\cdot\sec^2(\theta)\,d\theta$$

$$= \int \sec(\theta)\,d\theta$$

$$= \ln|\sec(\theta) + \tan(\theta)| + C$$

$$\boxed{= \ln|\sqrt{x^2+1} + x| + C}$$

When the integral involves the form $\sqrt{k^2-x^2}$, we use the substitution $x = k\sin(\theta)$. Then

$$\sqrt{k^2-x^2} = \sqrt{k^2-k^2\sin^2(\theta)} = \sqrt{k^2[1-\sin^2(\theta)]}$$

$$= \sqrt{k^2\cos^2(\theta)} = k\cos(\theta).$$

Also, $dx = k\cos(\theta)\,d\theta$.

$$\text{eg} \quad \int \frac{1}{x^2\sqrt{4-x^2}}\,dx$$

Here $k=2$ so we set $x = 2\sin(\theta)$ so $dx = 2\cos(\theta)\,d\theta$.

We have $\sqrt{4-x^2} = \sqrt{4-4\sin^2(\theta)}$

$$= \sqrt{4[1-\sin^2(\theta)]}$$

$$= \sqrt{4\cos^2(\theta)} = 2\cos(\theta)$$

$$x^2 = 4\sin^2(\theta)$$

The integral becomes

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{[4\sin^2(\theta)]\cdot[2\cos(\theta)]}\cdot 2\cos(\theta)\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sin^2(\theta)}\,d\theta$$

$$= \frac{1}{4}\int \csc^2(\theta)\,d\theta$$

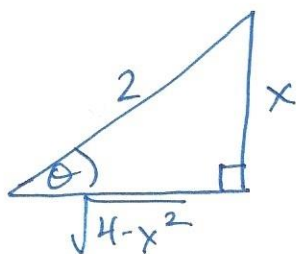$$= \frac{1}{4}\left[-\cot(\theta)\right] + C$$

$$\int \frac{1}{x^2 \sqrt{4-x^2}}\, dx = -\frac{1}{4}\cot(\theta) + C$$

We know that $\quad x = 2\sin(\theta)\quad$ so $\quad \sin(\theta) = \frac{1}{2}x$

$$\sqrt{4-x^2} = 2\cos(\theta)\quad$$ so $\quad \cos(\theta) = \frac{1}{2}\sqrt{4-x^2}$

and thus $\quad \cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{\frac{1}{2}\sqrt{4-x^2}}{\frac{1}{2}x} = \dfrac{\sqrt{4-x^2}}{x}$

Alternatively, we can use $\sin(\theta) = \frac{1}{2}x = \frac{x}{2}$ to create a right triangle with included angle θ, opposite side of length x, and hypotenuse of length 2.

By the Pythagorean theorem, the adjacent side is of length

$$\sqrt{4-x^2}\,.$$

Again, this tells us that $\cot(\theta) = \dfrac{\sqrt{4-x^2}}{x}$.

Either way, $\displaystyle \int \frac{1}{x^2\sqrt{4-x^2}}\, dx = -\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x} + C$

$$\boxed{= -\frac{\sqrt{4-x^2}}{4x} + C}$$

When the integral involves the form $\sqrt{x^2 - k^2}$, we use the substitution $x = k\sec(\theta)$. Then

$$\sqrt{x^2 - k^2} = \sqrt{k^2\sec^2(\theta) - k^2} = \sqrt{k^2\left[\sec^2(\theta) - 1\right]}$$
$$= \sqrt{k^2\tan^2(\theta)} = k\tan(\theta)\,.$$

Here, $dx = k \sec(\theta) \tan(\theta) d\theta$.

eg $\displaystyle\int \frac{\sqrt{x^2 - 5}}{x} dx$

Since $k = \sqrt{5}$, we set $x = \sqrt{5} \sec(\theta)$

$$dx = \sqrt{5} \sec(\theta) \tan(\theta) d\theta$$

and $\sqrt{x^2 - 5} = \sqrt{5 \sec^2(\theta) - 5} = \sqrt{5[\sec^2(\theta) - 1]}$

$$= \sqrt{5 \tan^2(\theta)} = \sqrt{5} \tan(\theta).$$

The integral becomes

$$\int \frac{\sqrt{x^2 - 5}}{x} dx = \int \frac{\sqrt{5} \tan(\theta)}{\sqrt{5} \sec(\theta)} \cdot \sqrt{5} \sec(\theta) \tan(\theta) d\theta$$

$$= \sqrt{5} \int \tan^2(\theta) d\theta$$

$$= \sqrt{5} \int [\sec^2(\theta) - 1] d\theta$$

$$= \sqrt{5} [\tan(\theta) - \theta] + C$$

$$= \sqrt{5} \tan(\theta) - \sqrt{5} \theta + C$$

$$= \sqrt{5} \cdot \frac{\sqrt{x^2 - 5}}{\sqrt{5}} - \sqrt{5} \cdot \text{arcsec}\left(\frac{\sqrt{5}x}{5}\right) + C$$

$$\boxed{= \sqrt{x^2 - 5} - \sqrt{5}\, \text{arcsec}\left(\frac{\sqrt{5}x}{5}\right) + C}$$

Since $\sec(\theta) = \frac{x}{\sqrt{5}} = \frac{\sqrt{5}x}{5}$

$\theta = \text{arcsec}\left(\frac{\sqrt{5}x}{5}\right)$

We can sometimes apply trigonometric substitution even when
an integral does not immediately appear to include a
square root.

$$eg \int \frac{x^3}{(x^2+4)^3} dx = \int \frac{x^3}{(\sqrt{x^2+4})^6} dx$$

We let $x = 2\tan(\theta)$ so $dx = 2\sec^2(\theta)d\theta$.

Then $\sqrt{x^2+4} = \sqrt{4\tan^2(\theta)+4} = 2\sec(\theta)$

$$(\sqrt{x^2+4})^6 = 64\sec^6(\theta)$$

$$x^3 = 8\tan^3(\theta)$$

The integral becomes

$$\int \frac{x^3}{(x^2+4)^3} dx = \int \frac{8\tan^3(\theta)}{64\sec^6(\theta)} \cdot 2\sec^2(\theta)d\theta$$

$$= \frac{1}{4} \int \frac{\tan^3(\theta)}{\sec^4(\theta)} d\theta$$

$$= \frac{1}{4} \int \frac{\sin^3(\theta)}{\cos^3(\theta)} \cdot \cos^4(\theta) d\theta$$

Let $u = \sin(\theta)$

$du = \cos(\theta)d\theta$

$$= \frac{1}{4} \int \sin^3(\theta) \cos(\theta) d\theta$$
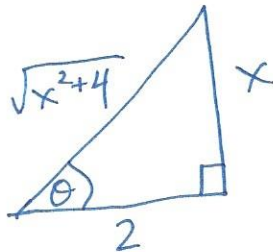
$$= \frac{1}{4} \int u^3 du$$

$$= \frac{1}{4} \left[\frac{u^4}{4} \right] + C$$

$$= \frac{1}{16} u^4 + C$$

Now we have

$$\int \frac{x^3}{(x^2+4)^3}\, dx = \frac{1}{16} \sin^4(\theta) + C$$

Since $x = 2\tan(\theta)$, $\tan(\theta) = \frac{x}{2}$ so we construct an appropriate right triangle:

Thus $\sin(\theta) = \dfrac{x}{\sqrt{x^2+4}}$ and so

$$\int \frac{x^3}{(x^2+4)^3}\, dx = \frac{1}{16} \cdot \left(\frac{x}{\sqrt{x^2+4}} \right)^4 + C$$

$$\boxed{= \frac{x^4}{16(x^2+4)^2} + C}$$