

Section 3.2: Trigonometric Integrals

Target: Integrals of powers of trigonometric functions and their products.

We will focus on integrals of the form $\int \sin^m(x) \cos^n(x) dx$.

① Suppose m is odd. We factor aside one $\sin(x)$ to serve as part of the differential in u -substitution. We then transform any remaining $\sin(x)$ into $\cos(x)$ using the identity $\sin^2(x) + \cos^2(x) = 1$, so $\sin^2(x) = 1 - \cos^2(x)$. Now we let $u = \cos(x)$ so $du = -\sin(x) dx$ and $-du = \sin(x) dx$.

$$\begin{aligned} \text{eg } & \int \sin^3(x) \cos^6(x) dx \\ &= \int \sin^2(x) \cos^6(x) \cdot \sin(x) dx \\ &= \int [1 - \cos^2(x)] \cos^6(x) \cdot \sin(x) dx \\ &= - \int [1 - u^2] u^6 du \\ &= - \int (u^6 - u^8) du \\ &= - \left[\frac{u^7}{7} - \frac{u^9}{9} \right] + C \end{aligned}$$

$$\begin{aligned} \text{Let } u &= \cos(x) \\ -du &= \sin(x) dx \end{aligned}$$

$$\boxed{= \frac{1}{9} \cos^9(x) - \frac{1}{7} \cos^7(x) + C}$$

② Suppose n is odd. We factor aside one $\cos(x)$ to serve as part of the differential in u -substitution. We then transform any remaining $\cos(x)$ into $\sin(x)$ using the identity $\cos^2(x) = 1 - \sin^2(x)$. Now we let $u = \sin(x)$ so $du = \cos(x) dx$.

$$\begin{aligned} \text{eg } & \int \sin^6(3x+1) \cos^5(3x+1) dx \\ &= \int \sin^6(3x+1) \cos^4(3x+1) \cdot \cos(3x+1) dx \\ &= \int \sin^6(3x+1) [\cos^2(3x+1)]^2 \cdot \cos(3x+1) dx \\ &= \int \sin^6(3x+1) [1 - \sin^2(3x+1)]^2 \cdot \cos(3x+1) dx \end{aligned}$$

We let $u = \sin(3x+1)$ so $du = 3\cos(3x+1) dx$
 $\frac{1}{3} du = \cos(3x+1) dx$

The integral becomes

$$\begin{aligned} \frac{1}{3} \int u^6 [1 - u^2]^2 du &= \frac{1}{3} \int (u^6 - 2u^8 + u^{10}) du \\ &= \frac{1}{3} \left[\frac{u^7}{7} - 2 \cdot \frac{u^9}{9} + \frac{u^{11}}{11} \right] + C \end{aligned}$$

$$\begin{aligned} &= \frac{1}{21} \sin^7(3x+1) - \frac{2}{27} \sin^9(3x+1) \\ &\quad + \frac{1}{33} \sin^{11}(3x+1) + C \end{aligned}$$

These strategies will still work even if only one of $\sin(x)$ or $\cos(x)$ is present, as long as its power is odd.

$$\text{eg } \int_0^{\pi/12} \cos^3(2x) dx = \int_0^{\pi/12} \cos^2(2x) \cdot \cos(2x) dx$$

$$\text{Let } u = \sin(2x)$$

$$du = 2\cos(2x) dx$$

$$\frac{1}{2} du = \cos(2x) dx$$

$$\text{When } x=0, u = \sin(0) = 0$$

$$x = \frac{\pi}{12}, u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$= \int_0^{\pi/12} [1 - \sin^2(2x)] \cdot \cos(2x) dx$$

$$= \frac{1}{2} \int_0^{1/2} [1 - u^2] du$$

$$= \frac{1}{2} \left[u - \frac{u^3}{3} \right]_0^{1/2}$$

$$= \frac{1}{2} \left[\left(\frac{1}{2} - \frac{1}{24} \right) - 0 \right]$$

$$= \frac{11}{48}$$

③ If both m and n are odd then we can follow the strategy of ① or ②. We normally choose the strategy in which u will be whichever of $\sin(x)$ or $\cos(x)$ that has the larger power.

④ Now suppose that both m and n are even. We apply the half-angle identities

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \quad \text{and} \quad \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\begin{aligned}
 \text{eg } \int \cos^2(x-3) dx &= \int \frac{1 + \cos(2 \cdot (x-3))}{2} dx \\
 &= \frac{1}{2} \int [1 + \cos(2x-6)] dx \\
 &= \frac{1}{2} \left[x + \frac{\sin(2x-6)}{2} \right] + C \\
 &= \frac{1}{2} x + \frac{1}{4} \sin(2x-6) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{eg } \int \sin^4(x) dx &= \int [\sin^2(x)]^2 dx \\
 &= \int \left[\frac{1 - \cos(2x)}{2} \right]^2 dx \\
 &= \int \frac{1 - 2\cos(2x) + \cos^2(2x)}{4} dx \\
 &= \frac{1}{4} \int \left[1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2} \right] dx \\
 &= \frac{1}{4} \int \left[\frac{3}{2} - 2\cos(2x) + \frac{1}{2} \cos(4x) \right] dx \\
 &= \frac{1}{4} \left[\frac{3}{2} x - 2 \cdot \frac{\sin(2x)}{2} + \frac{1}{2} \cdot \frac{\sin(4x)}{4} \right] + C \\
 &= \frac{3}{8} x - \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C
 \end{aligned}$$

Section 3.3: Trigonometric Substitution

Target: Integrals involving the radical forms

$$\sqrt{x^2 + k^2}, \sqrt{k^2 - x^2} \text{ or } \sqrt{x^2 - k^2} \text{ where}$$

k is assumed to be a positive constant.

Our approach is to eliminate the square root by replacing x with an appropriate expression in terms of a new variable θ .

eg $\sqrt{x^2 + 1}$

We let $x = \tan(\theta)$ so we have

$$\begin{aligned} x^2 + 1 &= \tan^2(\theta) + 1 \\ &= \sec^2(\theta) \end{aligned}$$

$$\begin{aligned} \text{Then } \sqrt{x^2 + 1} &= \sqrt{\sec^2(\theta)} \\ &= \sec(\theta) \end{aligned}$$

This is called a trigonometric substitution.