Section 3.1

eg $$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

Now we multiply both sides of the equation by the original denominator:

$$\frac{1}{(x-2)(x+2)} \cdot (x-2)(x+2) = \left(\frac{A}{x-2} + \frac{B}{x+2}\right) \cdot (x-2)(x+2)$$

$$1 = A(x+2) + B(x-2)$$

For $x=2$, we have

$$1 = A(2+2) + B(2-2)$$

$$1 = 4A \rightarrow A = 1/4$$

For $x=-2$, we have

$$1 = A(-2+2) + B(-2-2)$$

$$1 = -4B \rightarrow B = -1/4$$

Then we see that

$$\frac{1}{x^2-4} = \frac{1/4}{x-2} + \frac{-1/4}{x+2} \boxed{= \frac{1/4}{x-2} - \frac{1/4}{x+2}}$$

Now $$\int \frac{1}{x^2-4}\,dx = \int \left[\frac{1/4}{x-2} - \frac{1/4}{x+2}\right] dx$$

$$= \frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C$$

We can apply the method of partial fractions to a definite integral in the usual way.

eg $$\int_1^4 \frac{9x+5}{3x^2+x}\,dx$$

We decompose into partial fractions:

$$\frac{9x+5}{3x^2+x} = \frac{9x+5}{x(3x+1)} = \frac{A}{x} + \frac{B}{3x+1}$$

$$9x+5 = A(3x+1) + Bx$$

For $x=0$,

$$5 = A\cdot 1 + B\cdot 0 \rightarrow A=5$$

For $x=-1/3$,

$$2 = A\cdot 0 + B\cdot\left(-\tfrac{1}{3}\right)$$

$$2 = -\tfrac{1}{3}B \rightarrow B=-6$$

Hence $$\frac{9x+5}{3x^2+x} = \frac{5}{x} - \frac{6}{3x+1}$$

$$\int_1^4 \frac{9x+5}{3x^2+x}\,dx = \int_1^4 \left(\frac{5}{x} - \frac{6}{3x+1}\right)dx$$

$$= \Big[5\ln|x| - 2\ln|3x+1|\Big]_1^4$$

$$\boxed{= 7\ln(4) - 2\ln(13)}$$

We cannot apply the method of partial fractions directly to an improper rational function, but may do so after performing long division.

eg $\displaystyle\int \frac{x^4+4x-3}{x^3-x^2}\,dx$

$$\begin{array}{r l}
 & x+1 \\
x^3-x^2 \,\big)\!\! & \overline{x^4 \qquad\quad +4x-3} \\
 & x^4-x^3 \\
\hline
 & x^3 \quad +4x-3 \\
 & x^3-x^2 \\
\hline
 & x^2+4x-3
\end{array}$$

Now we can write $\dfrac{x^4+4x-3}{x^3-x^2} = x+1+\dfrac{x^2+4x-3}{x^3-x^2}$

and apply the method of partial fractions:

$$\frac{x^2+4x-3}{x^3-x^2} = \frac{x^2+4x-3}{x^2(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$$

$$x^2+4x-3 = Ax(x-1)+B(x-1)+Cx^2$$

For $x=0$, $-3=-B \rightarrow B=3$

For $x=1$, $2=C \rightarrow C=2$

For $x=-1$, $-6 = 2A-2B+C$

$-6 = 2A-6+2$

$-2=2A \rightarrow A=-1$

Now we can write

$$\int \frac{x^4+4x-3}{x^3-x^2}\,dx = \int \left((x+1) - \frac{1}{x} + \frac{3}{x^2} + \frac{2}{x-1} \right) dx$$

$$= \frac{x^2}{2} + x - \ln|x| + 3 \cdot \frac{x^{-1}}{-1} + 2\ln|x-1| + C$$

$$\boxed{= \frac{1}{2}x^2 + x - \ln|x| - \frac{3}{x} + 2\ln|x-1| + C}$$

The process of finding the constants in the partial fraction decomposition becomes more cumbersome with more repeated and/or irreducible quadratic factors of the denominator.

eg $\displaystyle\int \frac{7x^2-7x+1}{x^3-2x^2+x-2}\,dx$

Note that $x^3-2x^2+x-2 = x^2(x-2) + (x-2)$

$$= (x-2)(x^2+1)$$

so $$\frac{7x^2-7x+1}{x^3-2x^2+x-2} = \frac{7x^2-7x+1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

$$7x^2-7x+1 = A(x^2+1) + (Bx+C)(x-2)$$

For $x=2$, $\quad 15 = 5A \rightarrow A = 3$

$x=0$, $\quad 1 = 3 \cdot 1 + C \cdot (-2)$

$\quad -2 = -2C \rightarrow C = 1$

$x=1$, $\quad 1 = 3 \cdot 2 + (B+1) \cdot (-1)$

$\quad -5 = -B-1 \rightarrow B = 4$

Now we have

$$\int \frac{7x^2-7x+1}{x^3-2x^2+x-2}\,dx = \int \left(\frac{3}{x-2} + \frac{4x+1}{x^2+1} \right) dx$$

$$= \int \left(\frac{3}{x-2} + \frac{4x}{x^2+1} + \frac{1}{x^2+1} \right) dx$$

Let $u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$= 3\ln|x-2| + \arctan(x) + 4\int \frac{x}{x^2+1}\,dx$$

$$= 3\ln|x-2| + \arctan(x) + 2\int \frac{1}{u}\,du$$

$$= 3\ln|x-2| + \arctan(x) + 2\ln|u| + C$$

$$\boxed{= 3\ln|x-2| + \arctan(x) + 2\ln(x^2+1) + C}$$