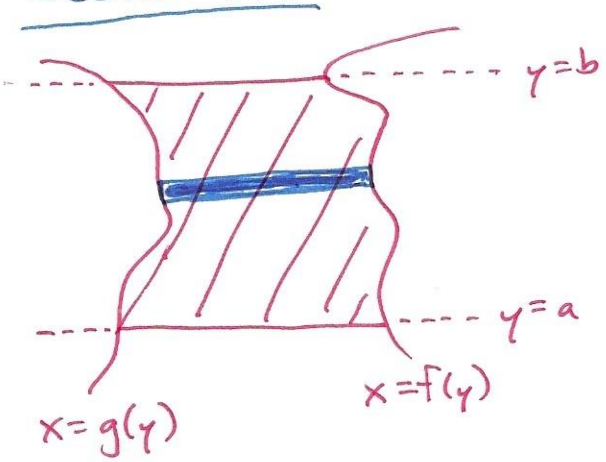
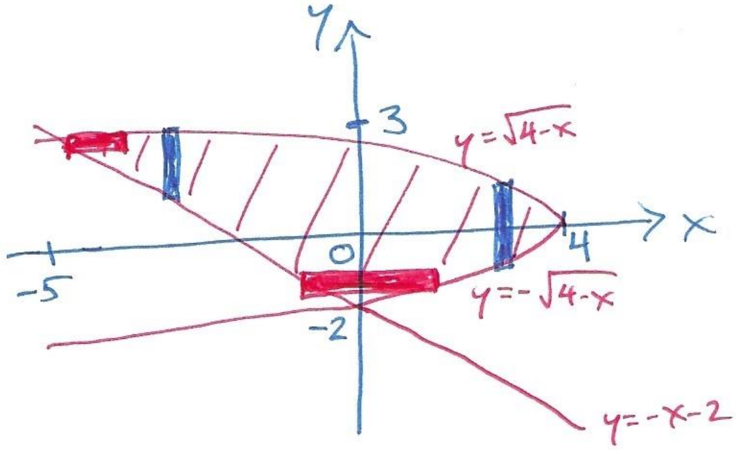
Section 2.4



Now consider a region bounded by a curve $x=f(y)$ to the right, $x=g(y)$ to the left, and the horizontal lines $y=a$ and $y=b$ on bottom and on top. Then this a <u>horizontally simple</u> region, and can be approximated using horizontally-oriented rectangles. Therefore the area A of this region is

$$A = \int_a^b [f(y) - g(y)]\,dy$$

eg Find the area of the region bounded by $y=-x-2$, $y=\sqrt{4-x}$ and $y=-\sqrt{4-x}$.



This region is not vertically simple, but can be divided into two vertically simple regions.

We need to find where

$$-x-2 = \sqrt{4-x}$$
$$(-x-2)^2 = 4-x$$
$$x^2+4x+4 = 4-x$$
$$x^2+5x = 0$$
$$x(x+5) = 0 \to \underset{\text{OMIT}}{x=0},\ x=-5$$

The first vertically simple region lies on $[-5, 0]$ with $f(x) = \sqrt{4-x}$ and $g(x) = -x-2$. Its area is

$$A_1 = \int_{-5}^{0} \left[\sqrt{4-x} - (-x-2) \right] dx$$

$$= \int_{-5}^{0} \left(\sqrt{4-x} + x + 2 \right) dx$$

$$= \left[\frac{(4-x)^{3/2}}{-3/2} + \frac{1}{2}x^2 + 2x \right]_{-5}^{0}$$

$$= \left[-\frac{2}{3}(4-x)^{3/2} + \frac{1}{2}x^2 + 2x \right]_{-5}^{0} = \frac{61}{6}$$

The second vertically simple region lies on $[0, 4]$ with $f(x) = \sqrt{4-x}$ and $g(x) = -\sqrt{4-x}$. Its area is

$$A_2 = \int_{0}^{4} \left[\sqrt{4-x} - (-\sqrt{4-x}) \right] dx$$

$$= 2\int_{0}^{4} \sqrt{4-x}\, dx$$

$$= 2\left[\frac{(4-x)^{3/2}}{-3/2} \right]_{0}^{4} = \frac{32}{3}$$

The area A of the entire region is $A = \frac{61}{6} + \frac{32}{3} \boxed{= \frac{125}{6}}$

We can rewrite $y = \pm\sqrt{4-x}$ as

$$y^2 = 4-x$$

$$x = 4-y^2$$

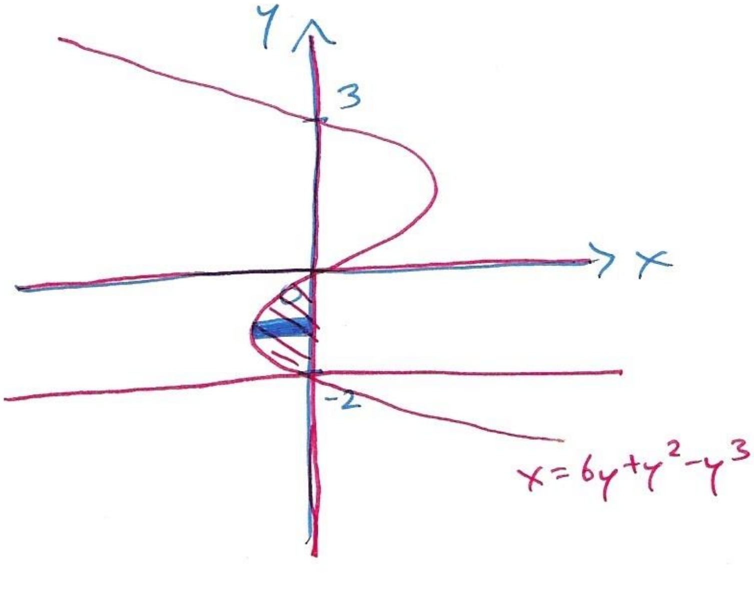
and $y=-x-2$ as $x=-y-2$. As a horizontally simple region, then, the region is bounded by $f(y)=4-y^2$ and $g(y)=-y-2$ on the y-interval $[-2,3]$. Hence

$$A = \int_{-2}^{3} \left[(4-y^2) - (-y-2)\right] dy$$

$$= \int_{-2}^{3} (6+y-y^2)\, dy$$

$$= \left[6y + \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_{-2}^{3} \quad \boxed{= \frac{125}{6}}$$

eg Find the area of the region bounded by $x = 6y+y^2-y^3$, $y=-2$, and the x- and y-axes.



We can write

$$x = 6y+y^2-y^3$$

$$= -y(y^2-y-6)$$

$$= -y(y-3)(y+2)$$

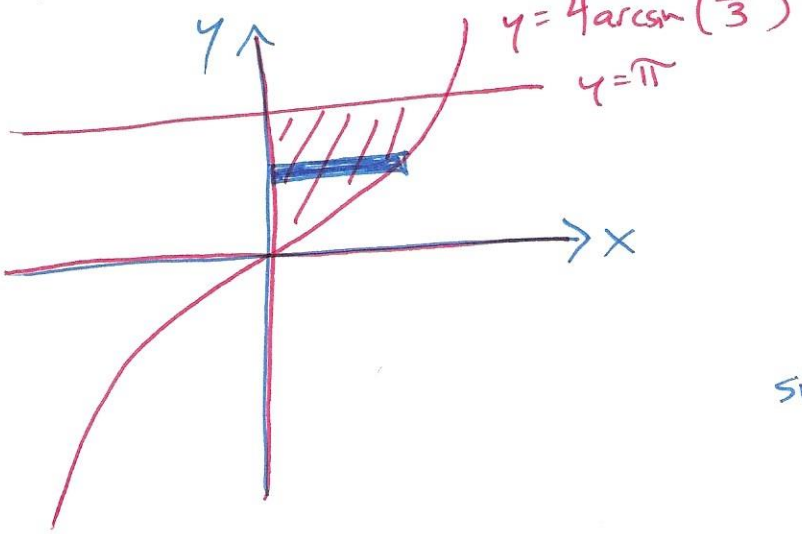
This is a horizontally simple region with $f(y)=0$, $g(y)=6y+y^2-y^3$ on $[-2,0]$.

$$A = \int_{-2}^{0} \left[0 - (6y + y^2 - y^3)\right] dy$$

$$= \int_{-2}^{0} (y^3 - y^2 - 6y)\, dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{3} - 3y^2\right]_{-2}^{0} \quad \boxed{= \frac{16}{3}}$$

eg Find the area of the region between $y = 4\arcsin\left(\frac{x}{3}\right)$, $y = \pi$, $y = 0$ and $x = 0$.



We can rewrite

$$y = 4\arcsin\left(\frac{x}{3}\right)$$

$$\frac{y}{4} = \arcsin\left(\frac{x}{3}\right)$$

$$\sin\left(\frac{y}{4}\right) = \frac{x}{3}$$

$$x = 3\sin\left(\frac{y}{4}\right)$$

While this region is both vertically and horizontally simple, it appears that the integration will be simpler in terms of y:

$$A = \int_{0}^{\pi} \left[3\sin\left(\frac{y}{4}\right) - 0\right] dy$$

$$= 3\int_{0}^{\pi} \sin\left(\frac{y}{4}\right) dy$$

$$= 3\left[\frac{-\cos(y/4)}{1/4}\right]_{0}^{\pi}$$

$$= -12\left[\cos\left(\frac{y}{4}\right)\right]_{0}^{\pi} \quad \boxed{= 12 - 6\sqrt{2}}$$