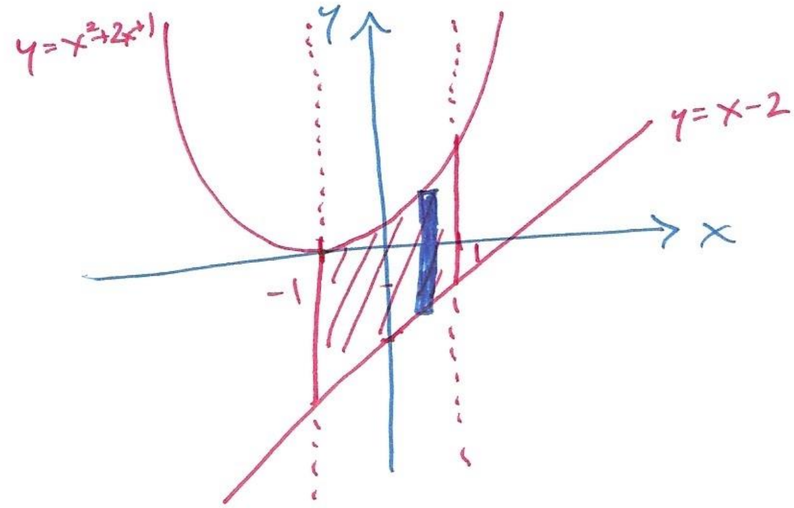
Section 2.4

$$A = \int_a^b [f(x) - g(x)]\,dx$$

eg Find the area of the region bounded by $y = x^2 + 2x + 1$ and $y = x - 2$ on the interval $[-1, 1]$.



Observe that

$$y = x^2 + 2x + 1 = (x+1)^2.$$

We can see that, on $[-1,1]$,

$$x^2 + 2x + 1 \geq x - 2$$

so $f(x) = x^2 + 2x + 1$

$g(x) = x - 2$.

Alternatively we could identify $f(x)$ and $g(x)$ by first setting

$$x^2 + 2x + 1 = x - 2$$

$$x^2 + x + 3 = 0 \quad \rightarrow \quad x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{-11}}{2}$$

This equation has no solution so the two curves have no points of intersection.

Thus we check a point like $x = 0$:

$$x^2 + 2x + 1 = 1 \qquad x - 2 = -2$$

so $x^2 + 2x + 1 \geq x - 2$ on $[-1, 1]$ so $f(x) = x^2 + 2x + 1$

$g(x) = x - 2$.

$$A = \int_{-1}^{1} \left[(x^2+2x+1) - (x-2)\right] dx$$

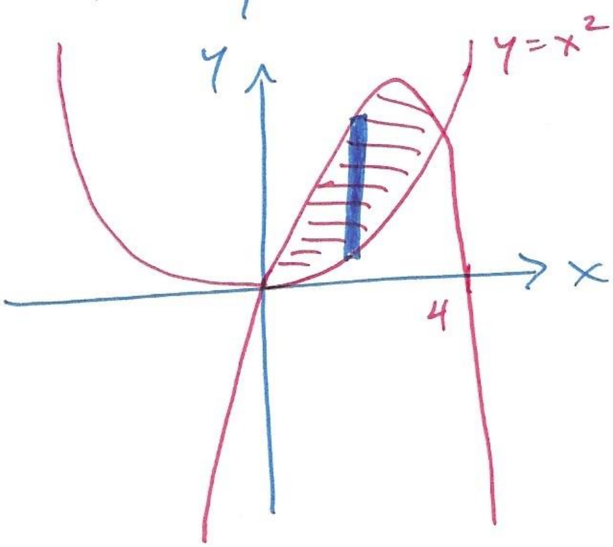
$$= \int_{-1}^{1} (x^2+x+3)\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 3x\right]_{-1}^{1}$$

$$= \left(\frac{1}{3}+\frac{1}{2}+3\right) - \left(\frac{-1}{3}+\frac{1}{2}-3\right) \boxed{= \frac{20}{3}}$$

Sometimes, the top and bottom boundary curves will form a natural region via their points of intersection, which means that we don't always have to specify an interval $[a, b]$.

eg Find the area of the region between $y = x^2$ and $y = 4x - x^2$.



We must solve for the points of intersection, because they will be the bounds on the definite integral for A.

We set

$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2x(x-2) = 0$$

$$x=0 \quad x=2$$

Thus we can see from the graph that, on $[0, 2]$,

$$f(x) = 4x - x^2$$

$$g(x) = x^2$$

$$A = \int_0^2 \left[(4x - x^2) - x^2 \right] dx$$

$$= \int_0^2 (4x - 2x^2)\, dx$$

$$= \left[4 \cdot \frac{x^2}{2} - 2 \cdot \frac{x^3}{3} \right]_0^2$$

$$= \left[2x^2 - \frac{2}{3}x^3 \right]_0^2$$

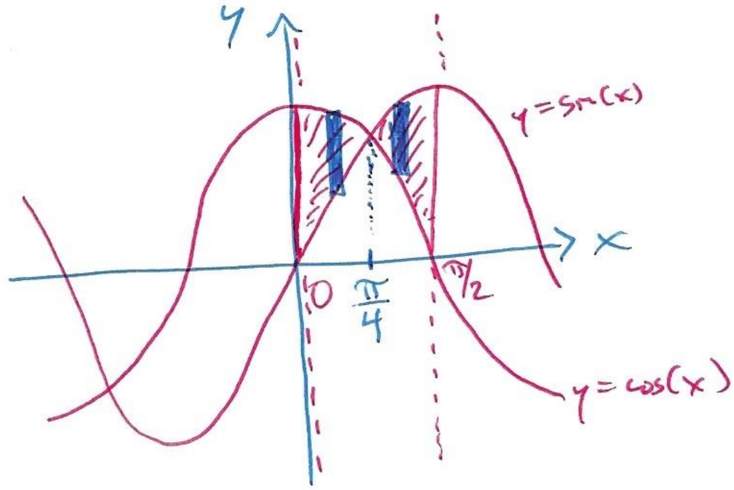
$$= \left[8 - \frac{16}{3} \right] - \left[0 - 0 \right] \boxed{= \frac{8}{3}}$$

The regions considered so far have been vertically simple. If we sketch a representative, vertically-oriented rectangle anywhere in the region, its top would always be defined by the same curve $y = f(x)$, and its bottom by the same curve $y = g(x)$.

What if a region isn't vertically simple?

One possibility is that we may be able to divide it into two or more regions that are vertically simple. Then we can use the area between curves formula to find the area of each vertically simple sub-region and add them together to get the area of the entire region.

eg Find the area of the region between $y = \sin(x)$ and $y = \cos(x)$ on $[0, \pi/2]$.



We set

$$\sin(x) = \cos(x)$$

$$\tan(x) = 1$$

$$x = \arctan(1) = \frac{\pi}{4}$$

So now we consider the vertically simple region on $[0, \pi/4]$ where $f(x) = \cos(x)$ and $g(x) = \sin(x)$.

Then the area A_1 of this region is

$$A_1 = \int_0^{\pi/4} [\cos(x) - \sin(x)]\, dx$$

$$= \Big[\sin(x) + \cos(x)\Big]_0^{\pi/4}$$

$$= \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] - [0 + 1] = \sqrt{2} - 1$$

Next we consider the vertically simple region on $[\pi/4, \pi/2]$ where $f(x) = \sin(x)$ and $g(x) = \cos(x)$.

Then the area A_2 of this region is

$$A_2 = \int_{\pi/4}^{\pi/2} [\sin(x) - \cos(x)]\, dx$$

$$A_2 = \Big[-\cos(x) - \sin(x)\Big]_{\pi/4}^{\pi/2}$$

$$= \Big[0 - 1\Big] - \Big[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\Big]$$

$$= -1 + \sqrt{2} \quad = \sqrt{2} - 1$$

The area A of the entire region on $[0, \pi/2]$ is

$$A = A_1 + A_2 = (\sqrt{2}-1) + (\sqrt{2}-1) \boxed{= 2\sqrt{2} - 2}.$$