Section 2.3

FTC ② : $\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Why does $F(x)$ need only be an antiderivative of $f(x)$ and not the indefinite integral? Observe that

$$\left[F(x) + C\right]_a^b = \left[F(b) + C\right] - \left[F(a) + C\right]$$
$$= F(b) + C - F(a) - C$$
$$= F(b) - F(a)$$

To apply integration by parts to a definite integral we just rewrite the integration by parts formula as

$$\int_a^b w\,dv = \left[vw\right]_a^b - \int_a^b v\,dw$$

eg $\displaystyle\int_1^3 \frac{\ln(x) - 1}{x^2}\,dx$

We use integration by parts with

$w = \ln(x) - 1$ $\qquad dw = \frac{1}{x}dx$

$dv = \frac{1}{x^2}dx$ $\qquad v = -\frac{1}{x}$

So $$\int_1^3 \frac{\ln(x) - 1}{x^2}\,dx = \left[-\frac{1}{x}(\ln(x) - 1)\right]_1^3 + \int_1^3 \frac{1}{x^2}dx$$
$$= \left[-\frac{\ln(x)}{x} + \frac{1}{x}\right]_1^3 + \left[-\frac{1}{x}\right]_1^3$$

$$= \left[-\frac{\ln(x)}{x} \right]_1^3$$

$$= -\frac{\ln(3)}{3} + \frac{\ln(1)}{1} \boxed{= -\frac{\ln(3)}{3}}$$

For a definite integral $\int_a^b f(x)\,dx$, the bounds a and b are values of x. When applying u-substitution, then, we must find the corresponding u-values for $x=a$ and $x=b$, and use them as the new bounds of integration. However, we do not need to revert everything back to x after integrating: FTC ② can be applied directly to the definite integral written in terms of u.

eg $\int_0^2 3x^2(x^3+1)^3\,dx$

Let $u = x^3+1$

$du = 3x^2\,dx$

Thus, when $x=0$, $u = 0^3+1 = 1$

$x=2$, $u = 2^3+1 = 9$

Now the integral becomes

$$\int_0^2 3x^2(x^3+1)^3\,dx = \int_1^9 u^3\,du$$

$$= \left[\frac{u^4}{4}\right]_1^9 = \frac{9^4}{4} - \frac{1^4}{4} \boxed{= 1640}$$

eg $\int_0^{\pi/3} \sec^6(\theta)\tan(\theta)\,d\theta$

$= \int_0^{\pi/3} \sec^5(\theta) \cdot \sec(\theta)\tan(\theta)\,d\theta$

Let $u = \sec(\theta)$

$du = \sec(\theta)\tan(\theta)\,d\theta$

When $\theta = 0$, $u = \sec(0) = 1$

$\theta = \frac{\pi}{3}$, $u = \sec\left(\frac{\pi}{3}\right) = 2$

The integral becomes

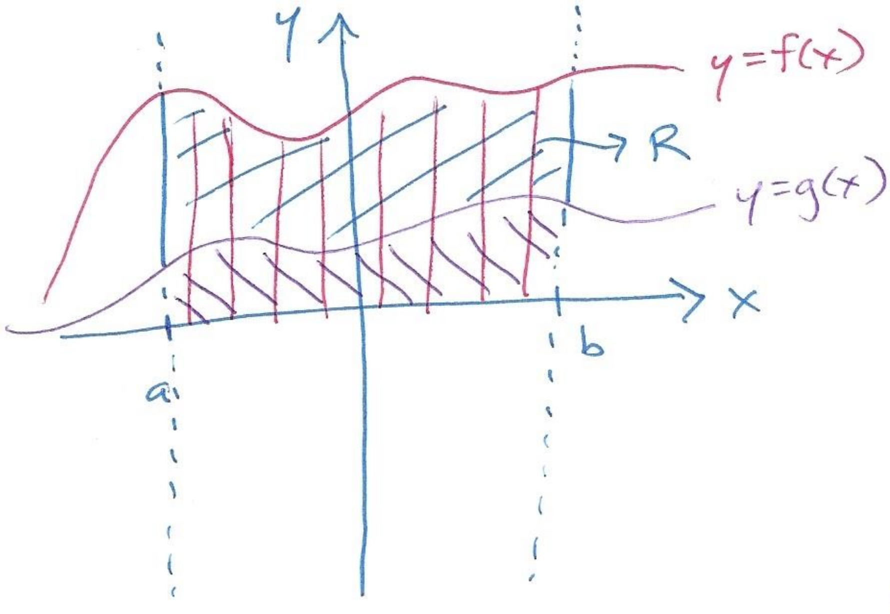
$$\int_0^{\pi/3} \sec^6(\theta)\tan(\theta)\,d\theta = \int_1^2 u^5\,du$$

$$= \left[\frac{u^6}{6}\right]_1^2$$

$$= \frac{2^6}{6} - \frac{1^6}{6} \boxed{= \frac{21}{2}}$$

Section 2.4: Area Between Curves

Now we will try to find the area A of a region R which is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, to the left by $x = a$, and to the right $x = b$.



Let R_1 be the region under $y = f(x)$ on $[a, b]$, and let A_1 be its area.

Let R_2 be the region under $y = g(x)$ on $[a, b]$, and let A_2 be its area.

Then
$$\begin{aligned} A &= A_1 - A_2 \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b [f(x) - g(x)]\,dx \end{aligned}$$

This is the formula for the area between curves. Note that we must have both $f(x)$ and $g(x)$ continuous on $[a, b]$ and $f(x) \geq g(x)$ on $[a, b]$.

Note that, if $g(x) \equiv 0$ then the bottom boundary curve is the line $y = 0$, the x-axis. Then the area between curves formula becomes
$$A = \int_a^b [f(x) - 0]\,dx = \int_a^b f(x)\,dx$$
where $f(x) \geq 0$ on $[a, b]$. This is the area under a curve formula again.

Given two boundary curves, how do we identify $f(x)$ and $g(x)$?

1. Sketch the graph.
2. Check for any points of intersection on $[a,b]$ by setting the functions equal to each other.
   If there are none, choose any point $x=p$ on $[a,b]$ and evaluate the functions there.
   The function that gives the larger value is $f(x)$.
   The other is $g(x)$.