Section 2.3

Theorem: The Second Fundamental Theorem

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: Recall that the Mean Value Theorem (MVT) states that if $z(x)$ is continuous and differentiable on $[c,d]$ then there exists a point $x=p$ on (c,d) for which

$$z'(p) = \frac{z(d)-z(c)}{d-c}.$$

Now observe that $F(x)$ satisfies the MVT on $[a,b]$ and, therefore, on any of the n subintervals into which $[a,b]$ can be partitioned.

So consider one such subinterval $[x_{i-1}, x_i]$. By the MVT, there exists a point $x=p_i$ on (x_{i-1}, x_i) for which

$$F'(p_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

$$= \frac{F(x_i) - F(x_{i-1})}{\Delta x_i}$$

Furthermore, since $F(x)$ is an antiderivative of $f(x)$ then $F'(x) = f(x)$. So now we can write

$$f(p_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x_i}$$

$$f(p_i)\Delta x_i = F(x_i) - F(x_{i-1})$$

Now recall that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

If we set the sample point $x_i^* = p_i$ then we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(p_i)\Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1})\right]$$

Observe that

$$\sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1})\right]$$

$$= \left[F(x_1) - F(x_0)\right] + \left[F(x_2) - F(x_1)\right]$$

$$+ \left[F(x_3) - F(x_2)\right] + \left[F(x_4) - F(x_3)\right]$$

$$+ \cdots + \left[F(x_{n-1}) - F(x_{n-2})\right] + \left[F(x_n) - F(x_{n-1})\right]$$

$$= F(x_n) - F(x_0)$$

$$= F(b) - F(a)$$

Now we can write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[F(b)-F(a)\right]$$

$$= F(b) - F(a).$$

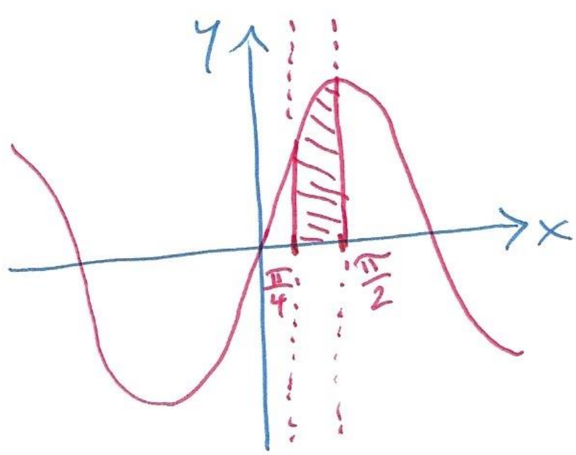
eg $$\int_0^{3/2} \frac{1}{\sqrt{3-x^2}}\,dx = \left[\arcsin\left(\frac{x}{\sqrt{3}}\right)\right]_0^{3/2}$$

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(0)$$

$$= \frac{\pi}{3} - 0 \quad \boxed{= \frac{\pi}{3}}$$

We can apply FTC 2 to find areas as well.

eg Find the area under $y = \sin(x)$ from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$.



$$A = \int_{\pi/4}^{\pi/2} \sin(x)\,dx$$

$$= \left[-\cos(x)\right]_{\pi/4}^{\pi/2}$$

$$= -\cos\left(\frac{\pi}{2}\right) - \left[-\cos\left(\frac{\pi}{4}\right)\right]$$

$$= 0 + \frac{\sqrt{2}}{2}$$

$$\boxed{= \frac{\sqrt{2}}{2}}$$

We can apply all of the methods for computing indefinite integrals to the evaluation of definite integrals using FTC ②.

eg $\int_0^4 \sqrt{x}\,(x-2)\,dx$

$$= \int_0^4 \left(x^{3/2} - 2x^{1/2}\right) dx$$

$$= \int_0^4 x^{3/2}\,dx - 2\int_0^4 x^{1/2}\,dx$$

$$= \left[\frac{x^{5/2}}{5/2}\right]_0^4 - 2\left[\frac{x^{3/2}}{3/2}\right]_0^4$$

$$= \frac{2}{5}\left[4^{5/2} - 0^{5/2}\right] - \frac{4}{3}\left[4^{3/2} - 0^{3/2}\right]$$

$$= \frac{2}{5}\left[32 - 0\right] - \frac{4}{3}\left[8 - 0\right]$$

$$= \frac{64}{5} - \frac{32}{3} \boxed{= \frac{32}{15}}$$

eg $\int_1^7 \frac{x+3}{3x}\,dx$

$$= \int_1^7 \left(\frac{1}{3} + \frac{1}{x}\right) dx$$

$$= \left[\frac{1}{3}x + \ln|x|\right]_1^7$$

$$= \left[\frac{7}{3} + \ln(7)\right] - \left[\frac{1}{3} + \ln(1)\right]$$

$$\boxed{= 2 + \ln(7)}$$

Given a definite integral $\int_a^b f(x)\,dx$ for which $f(x)$ is discontinuous on $[a,b]$, we call it an <u>improper</u> integral and FTC ② does <u>not</u> apply.

eg $\int_{-1}^{7} \frac{x+3}{3x}\,dx$ is improper because $\frac{x+3}{3x}$ is discontinuous at $x=0$, and $x=0$ lies on $[-1,7]$.

Sometimes we combine FTC ② with the Additive Interval Property.

eg $\int_1^6 |4-x|\,dx$ Recall $|4-x| = \begin{cases} 4-x, & x \le 4 \\ -(4-x), & x > 4 \end{cases}$

$$= \int_1^4 |4-x|\,dx + \int_4^6 |4-x|\,dx$$

$$= \int_1^4 (4-x)\,dx + \int_4^6 -(4-x)\,dx$$

$$= \left[4x - \frac{x^2}{2}\right]_1^4 - \left[4x - \frac{x^2}{2}\right]_4^6$$

$$= \left[(16-8) - \left(4 - \tfrac{1}{2}\right)\right] - \left[(24-18) - (16-8)\right]$$

$$\boxed{= \frac{13}{2}}$$