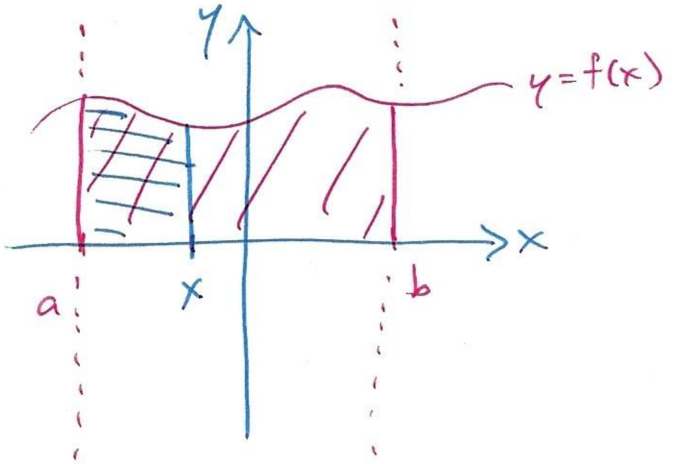
Section 2.3: The Fundamental Theorem of Calculus

Recall that if $f(x)$ is continuous and non-negative on $[a,b]$ then the area A of the region under the curve is given by

$$A = \int_a^b f(x)\,dx.$$

We could rewrite this in terms of any other variable of integration, such as

$$A = \int_a^b f(t)\,dt.$$



Now consider a point x on $[a,b]$. Then the area A_1 of the region under $y=f(x)$ on $[a,x]$ is given by

$$A_1 = \int_a^x f(t)\,dt.$$

Then this defines a function of x, which can be written

$$g(x) = \int_a^x f(t)\,dt.$$

For instance, in optics we study Fresnel functions like

$$S(x) = \int_0^x \sin(t^2)\,dt.$$

Now suppose we have a small constant h and consider both $g(x)$ and $g(x+h)$. Then $g(x+h)-g(x)$ represents the area under $y=f(x)$ which lies on the interval $[a, x+h]$ but not on $[a,x]$. This can be approximated as a rectangle of width h and height $f(x)$. Thus

$$g(x+h) - g(x) \approx hf(x)$$

$$\frac{g(x+h)-g(x)}{h} \approx f(x)$$

This approximation will become more and more accurate as $h \to 0$, so

$$f(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$$

<u>Theorem: The First Fundamental Theorem (FTC ①)</u>

If $f(t)$ is continuous on $[a,b]$ then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and differentiable on (a,b).

Furthermore, $$g'(x) = f(x).$$

eg Given $S(x) = \int_0^x \sin(t^2)\,dt$

$$S'(x) = \sin(x^2).$$

If the upper bound is a function of x, rather than just x itself, we apply FTC ① with the Chain Rule.

eg $g(x) = \int_1^{3x^2} e^{t^4}\,dt$

$$g'(x) = e^{(3x^2)^4} \cdot [3x^2]'$$

$$= e^{81x^8} \cdot 6x$$

$$\boxed{= 6x\,e^{81x^8}}$$

eg $y = \int_{-2}^{\ln(x)} \cos(t^3)\,dt$

$$\frac{dy}{dx} = \cos(\ln^3(x)) \cdot \frac{d}{dx}[\ln(x)]$$

$$\boxed{= \frac{\cos(\ln^3(x))}{x}}$$

Given a function of the form

$$g(x) = \int_x^a f(t)\,dt$$

then we apply FTC ① by rewriting it as

$$g(x) = -\int_a^x f(t)\,dt.$$

eg
$$g(x) = \int_x^3 \arcsin\left(\frac{1}{t^2}\right)dt$$

$$= -\int_3^x \arcsin\left(\frac{1}{t^2}\right)dt$$

$$g'(x) = -\left[\int_3^x \arcsin\left(\frac{1}{t^2}\right)dt\right]'$$

$$\boxed{= -\arcsin\left(\frac{1}{x^2}\right)}$$

Given a function of the form

$$g(x) = \int_{a(x)}^{b(x)} f(t)\,dt$$

we apply the Additive Interval Property and rewrite it as

$$g(x) = \int_{a(x)}^{0} f(t)\,dt + \int_0^{b(x)} f(t)\,dt$$

$$= -\int_0^{a(x)} f(t)\,dt + \int_0^{b(x)} f(t)\,dt$$

eg $g(x) = \int_x^{x^4} \tan(\sqrt{t})\,dt$

$= \int_x^0 \tan(\sqrt{t})\,dt + \int_0^{x^4} \tan(\sqrt{t})\,dt$

$= -\int_0^x \tan(\sqrt{t})\,dt + \int_0^{x^4} \tan(\sqrt{t})\,dt$

$g'(x) = -\tan(\sqrt{x}) + \tan(\sqrt{x^4}) \cdot [x^4]'$

$\boxed{= -\tan(\sqrt{x}) + 4x^3 \tan(x^2)}$

Theorem: The Second Fundamental Theorem (FTC ②)

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

eg $\int_{-2}^{1} x^3\,dx$

An antiderivative of x^3 is $\frac{x^4}{4}$ so, by FTC ②,

$$\int_{-2}^{1} x^3\,dx = \frac{1^4}{4} - \frac{(-2)^4}{4} = \frac{1}{4} - 4 \boxed{= -\frac{15}{4}}$$

Notationally, we can write

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$