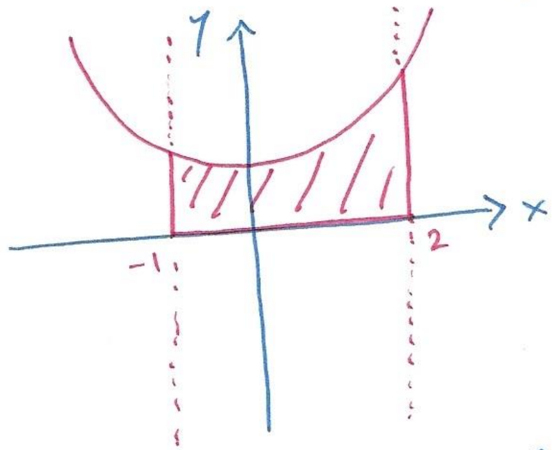
Section 2.1

eg (cont.) Find the area under $f(x) = x^2+1$, above the x-axis, and between $x=-1$ and $x=2$.



$$\Delta x = \frac{3}{n}$$

$$x_i^* = -1 + \frac{3i}{n}$$

$$f(x_i^*) = \frac{9i^2}{n^2} - \frac{6i}{n} + 2$$

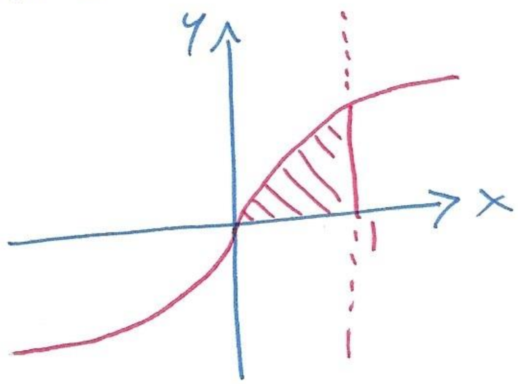
Then

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{6i}{n} + 2\right)\cdot\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27i^2}{n^3} - \frac{18i}{n^2} + \frac{6}{n}\right) \\
&= \lim_{n\to\infty} \left[\frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{6}{n}\sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty} \left[\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{18}{n^2}\cdot\frac{n(n+1)}{2} + \frac{6}{n}\cdot n\right] \\
&= \lim_{n\to\infty} \left[\frac{9(n+1)(2n+1)}{2n^2} - \frac{9(n+1)}{n} + 6\right] \\
&= 9 - 9 + 6 \\
&\boxed{= 6}
\end{aligned}$$

We can generalize the area formula to allow for an irregular partition, where each rectangle or subinterval has its own width Δx_i. Then

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i .$$

eg Find the area under $f(x) = \sqrt[3]{x}$ on $[0,1]$.



What if we use a regular partition?

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i^* = x_i = 0 + \frac{i}{n} = \frac{i}{n}$$

$$f(x_i^*) = \sqrt[3]{\frac{i}{n}}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\frac{i}{n}} \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^{1/3}}{n^{4/3}}$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^{4/3}} \sum_{i=1}^{n} i^{1/3} \right]$$

But we have no summation formula for $\sum_{i=1}^{n} i^{1/3}$

Instead, suppose we choose a partition for which the right endpoint of each subinterval is given by

$$x_i = \frac{i^3}{n^3}$$

Then $f(x_i^*) = f(x_i) = \sqrt[3]{\frac{i^3}{n^3}} = \frac{i}{n}$

Because we are now using an irregular partition, we cannot use the formula $\Delta x = \frac{b-a}{n}$. Instead, we have

$$\begin{aligned}
\Delta x_i &= x_i - x_{i-1} \\
&= \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} \\
&= \frac{i^3}{n^3} - \frac{i^3 - 3i^2 + 3i - 1}{n^3} \\
&= \frac{3i^2 - 3i + 1}{n^3}
\end{aligned}$$

Now

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{3i^2 - 3i + 1}{n^3} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i^3}{n^4} - \frac{3i^2}{n^4} + \frac{i}{n^4} \right) \\
&= \lim_{n\to\infty} \left[\frac{3}{n^4} \sum_{i=1}^{n} i^3 - \frac{3}{n^4} \sum_{i=1}^{n} i^2 + \frac{1}{n^4} \sum_{i=1}^{n} i \right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n^4} \cdot \frac{n^2(n+1)^2}{4} - \frac{3}{n^4} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4} \cdot \frac{n(n+1)}{2} \right] \\
&= \lim_{n\to\infty} \left[\frac{3(n+1)^2}{4n^2} - \frac{(n+1)(2n+1)}{2n^3} + \frac{n+1}{2n^3} \right] \\
&= \frac{3}{4} - 0 + 0 \boxed{= \frac{3}{4}}
\end{aligned}$$

The expression $\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$ is called a <u>Riemann sum</u>.

Section 2.2: Definite Integration

<u>Def'n</u>: If $f(x)$ is defined on a closed interval $[a,b]$ and $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ exists then we say that $f(x)$ is <u>integrable</u> on $[a,b]$.
Furthermore, we write

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

and we call this the <u>definite integral</u> of $f(x)$ on $[a,b]$. Here, $x=a$ is the <u>lower bound</u> of the definite integral and $x=b$ is its <u>upper bound</u>.

eg Given $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2 - e^{x_i})\Delta x$ on $[0,4]$

we can rewrite the limit of the Riemann sum as the definite integral

$$\int_0^4 (x^2 - e^x)\,dx.$$